IUMPST: The Journal. Vol 1 (Content Knowledge), September 2016 [www.k-12prep.math.ttu.edu]
ISSN 2165-7874

Secondary Pre-Service Teachers' Algebraic Reasoning About Linear Equation Solving

Christina Alvey
Mount Saint Mary College
330 Powell Avenue
Newburgh, NY 12550
Christina.Alvey@msmc.edu

Rick A. Hudson
University of Southern Indiana
8600 University Blvd.
Evansville, IN 47712
rhudson@usi.edu

Jill Newton
Purdue University
100 N. University St.
West Lafayette, IN 47907
janewton@purdue.edu

Lorraine M. Males
University of Nebraska-Lincoln
214A Henzlik Hall
Lincoln, NE 68588
lmales2@unl.edu

Abstract

This study analyzes the responses of 12 secondary pre-service teachers on two tasks focused on reasoning when solving linear equations. By documenting the choices PSTs made while engaging in these tasks, we gain insight into how new teachers work mathematically, reason algebraically, communicate their thinking, and make pedagogical decisions. We will share qualitative results from our examination of teacher knowledge through pre-service teachers' explanations, models, language, and conjectures about student thinking.

Introduction

Mathematics education researchers have begun to describe the knowledge that mathematics teachers utilize in their professional work. The majority of this research has examined the mathematical knowledge for teaching elementary mathematics (e.g., Ball, Thames, & Phelps, 2008), with fewer studies examining the knowledge for teaching secondary mathematics. Thus, there are many facets of secondary teacher knowledge and preparation left unexplored. Prior studies of research on secondary mathematics teacher knowledge have focused on student thinking (e.g., Asquith, Stephens, Knuth, & Alibali, 2007), differentiation between "knowing that" and "knowing why" (e.g., Even & Tirosh, 1995), and the link between mathematics content knowledge and mathematics teaching (e.g., McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012).

Built on Shulman's (1986) seminal work on pedagogical content knowledge, Ball et al. (2008) developed the Mathematical Knowledge for Teaching (MKT) framework, characterizing six domains of teachers' knowledge. Ball and colleagues argued that the knowledge needed to teach mathematics goes beyond the Common Content Knowledge (CCK) that is needed for many professions. Teachers must also deeply understand and be able to unpack that mathematics (Specialized Content Knowledge or SCK), be aware of the ways in which mathematical topics are connected across the curriculum (Horizon Content Knowledge or HCK), understand how students might approach problems and where they might find challenges (Knowledge of Content and Students or KCS), be familiar with the types of pedagogical strategies that might help students learn particular content (Knowledge of Content and Teaching or KCT), and have experiences with a range of instructional materials for teaching mathematics (Knowledge of Content and Curriculum or KCC).

Ball et al.'s (2008) framework provides important groundwork for mapping the knowledge associated with teaching, however their initial conceptualization was focused on the knowledge of elementary teachers. The knowledge that secondary teachers need is much different as their work requires a heavier focus on proof and axiomatic systems, attention to mathematical structure, and involvement of more conceptually-challenging mathematical content (Kilpatrick, Blume, Heid, Wilson, Wilson, & Zbiek, 2015). Kilpatrick et al.'s (2015) Mathematical Understanding for Secondary Teaching (MUST) framework includes mathematical activity such as justifying, proving, and reasoning when conjecturing and generalizing, but it also addresses the mathematical context of teaching, such as recognizing how students think about particular mathematical ideas.

To extend the literature on MKT, our work focuses on the algebraic reasoning of secondary pre-service teachers (PSTs) by documenting what choices they made in the act of doing algebra, including what they said, wrote, and drew; how they reflected on students' algebraic solutions; and how they discussed their pedagogical decisions related to solving equations. We approached this work with attention not only to the knowledge and actions of our participants, but also thinking about the use of the MKT framework for studies of those preparing to teach secondary mathematics. Although we agree with Speer, King and Howell (2015) that there are challenges when conceptualizing particular MKT domains for secondary and college mathematic teaching, we approached this work with attention not only to the knowledge and actions of our participants, but also used the MKT framework when designing the study and analyzing our PSTs' responses.

We investigated the pre-service teachers' understanding of the "Reasoning with Equations and Inequalities" (A-REI) domain of the *Common Core State Standards for Mathematics* (*CCSSM*) (National Governors Association [NGA] Center for Best Practices & Council of Chief State School Officers [CCSSO], 2010), which includes expectations related to solving, representing, and reasoning with a range of equation types. In this paper, we present our findings related to solving linear equations. Extant research focusing on teaching linear equation solving has addressed the use of a balance as an embodiment for teaching solution strategies and the use of realistic word problems (Andrews & Sayers, 2012), the affordances and challenges of shifting the conception of equation from a statement about unknown numbers to a comparison of two functions (Chazan, Yerushalmy, & Leikin, 2008), and the importance of encouraging perseverance in problem solving and flexible solution strategies (e.g., Huntley, Marcus, Kahan, & Miller, 2007; Star & Rittle-Johnson, 2008; Stueben & Torbert, 2006).

We were also interested in the language and explanations used by the PSTs as they solved equations. Participation in mathematical discourse (we use this term broadly to include verbal, graphical, symbolic, and pictorial communication) has been found to be challenging for both students and teachers (e.g., Arcavi, 1994; Pimm, 1987; Herbel-Eisenmann, & Otten, 2011; Stein, Engle, Smith, & Hughes, 2008); however, such communication is essential for productive mathematics classrooms and is recognized in policy recommendations as a key mathematical process (NCTM, 2000; NGA & CCSSO, 2010). In fact, some scholars conceptualize mathematics as a discourse and learning mathematics as a change in participation in that discourse (e.g., Sfard, 2008).

We sought to identify what the PSTs *know* and *are able to do* as they solve linear equations; our study is consequently focused on the following research questions:

1. What explanations do the PSTs use as they solve linear equations?
2. What language do the PSTs use as they engage in solving linear equations?
3. What benefits and limitations do the PSTs identify about algebraic models used in solving linear equations, such as the pan balance?

Research Methods & Data Sources

To investigate PSTs' knowledge of equations and inequalities, we conducted semi-structured, task-based interviews with 12 PSTs from a large, Midwestern university. In the current article, we focus on PSTs' responses to two items that addressed solving linear equations. The second item was adapted from Floden, Ferrini-Mundy, Senk, Reckase, & McCrory (2010).

Item 1:

a. Solve $5x + 9 = 2 + 7(x - 3)$. As you do so, please explain each step as you would to a high school student who is struggling with equation solving.

b. If a student asked why you can do the same thing to both sides, what would you say to them?

Item 2: Some textbooks suggest that teachers use a pan balance to represent mathematical sentences. For instance, if B represents the weight of each box pictured below (in ounces), and  represents a one-ounce weight, the balance pictured below represents the equation 3B + 4 = 10. (See Figure 1.)

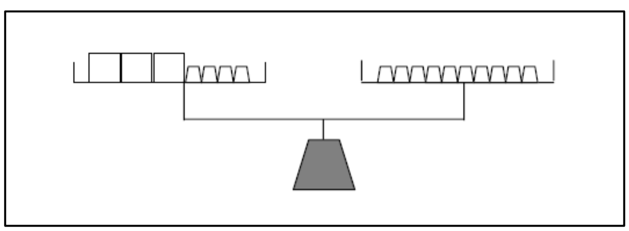

Figure 1.
Pan balance from Item 2 adapted from Floden, et al. (2010).

a. How can you use the pan balance to solve this equation?

b. What are some of the benefits of using the pan balance as a model for solving equations?

c. What are the limitations that might arise when using the pan balance as a model for solving equations?

We identified the types of knowledge in the MKT framework addressed by each of the five sub-questions in the two items (see Figure 2). The task-based interview protocol included questions that foregrounded the pre-service teachers' CCK, SCK, KCS, and KCT. Although the mapping between individual interview questions and domains of teacher knowledge is complex because the boundaries of the domains of knowledge are not easy to discern (Ball, et al., 2008), we have provided a preliminary mapping to assist the reader in seeing how we categorized the items. We identified Item 1a as a CCK item, because we anticipate that solving and explaining one's thinking when solving a linear equation is a skill that could be used in settings outside of the act of teaching. We also categorized Item 1a as KCS, as it requires one to understand and anticipate how students will react to a problem. Item 1b relates more to responding to students' mathematical inquiries in ways that build on the students' thinking and addresses their likely misconceptions (SCK), and requires one to think of new or different ways to describe the conceptual underpinnings of equation solving. Item 2a most closely aligns with SCK, as drawing connections between the pan balance and the act of solving equations is a mathematical skill that is unique to the act of teaching. Items 2b and 2c are KCT items, because they required the PSTs to "evaluate the instructional advantages or disadvantages of representations used to teach a specific idea" (Ball, et al., 2008, p. 401).

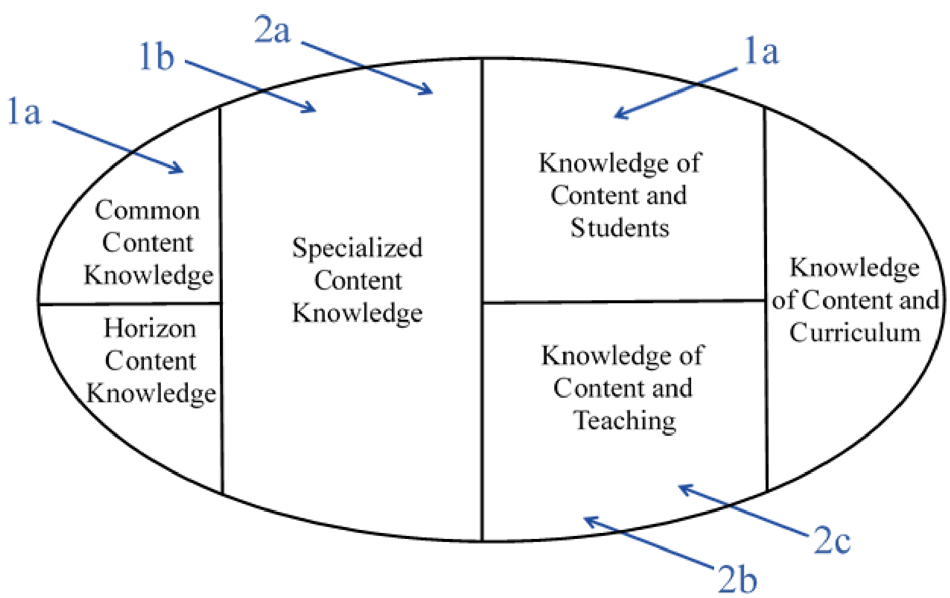

Figure 2.
Task alignment to Mathematical Knowledge for Teaching Framework adapted from Ball et al. (2008).

Participants were video recorded as they responded to items, and their written work was collected. Each video recording was transcribed verbatim, and each transcript was divided into sections of text that typically corresponded to one sentence. In some cases a participant did not complete a sentence or used run-on sentences, and a single section of text that included two sentences or a run-on sentence was split into two sections.

Two members of the research team coded each section of text using emergent coding techniques (Glaser & Strauss, 1967). The codes assigned were not intended to compare the PSTs explicitly, but rather to gain insight about what the PSTs knew and what they did in response to the interview items. The team members collaborated to ensure consistency in coding procedures and came to consensus when necessary. Many sections of text required the use of multiple codes, as the section addressed more than one of the research questions. The reconciled codes were organized into categories through a process of axial coding (Strauss & Corbin, 1998). In addition, the number of PSTs whose statements were included in the axial codes were tallied. For example, as we analyzed the explanations provided by PSTs for Item 1, there were five unique open codes that initially emerged (e.g., doing the same thing to both sides, what you do to one side you do to the other, keeping balance, doing something to both sides reminds students of two sides, doing something to both sides is the same as adding 0 on one side). The number of occurrences of these codes varied from one instance to four unique instances. As we sorted and categorized the open codes, these codes were combined into a single code termed "doing the same to both sides."

Finally, the PSTs' written work was analyzed qualitatively through open coding techniques. Nine codes were developed for Item 1a: showing distribution, using arrows to signify distribution, collection of x-terms, collection of constants, division by coefficient, checking work, writing x on the left-hand side (LHS), identifying like terms (e.g., by underlining or circling), showing work to justify operating on both sides of the equation. Three additional codes were developed for Item 1b: using equations, writing words, and using a pan balance. In general, PSTs responded to Item 1b verbally. A sample of PST Gabe's coded work for Item 1a is shown in Figure 3.

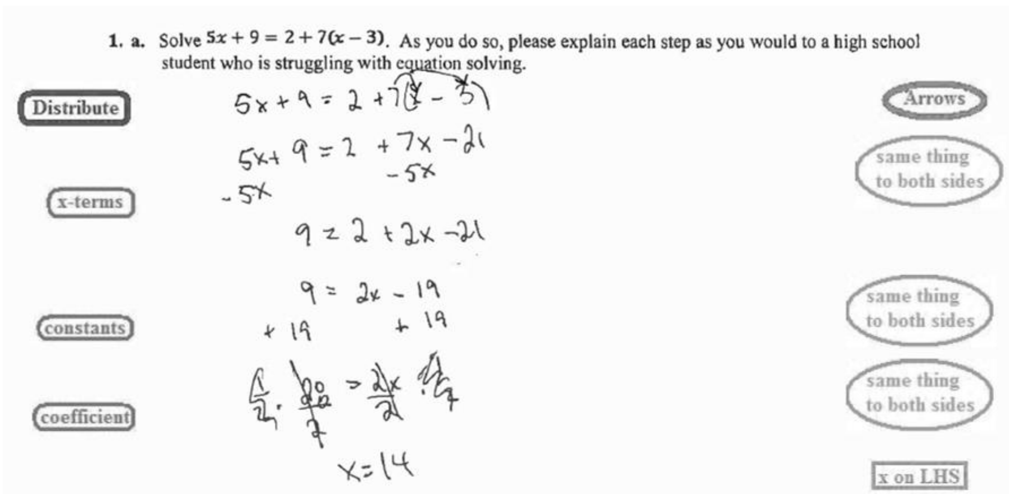

Figure 3.
PST Gabe's coded work.

Five codes were developed to analyze written work on Item 2: checking work, showing work to justify operating on both sides of the equation, showing symbolic algebra, crossing out small trapezoids, and crossing out large squares.

Findings

After reconciliation, 1192 uniquely-coded verbal statements and 117 uniquely-coded written statements/drawings were documented across the 12 interviews. In the sections below, each research question is addressed, and frequencies for the final codes are presented. In each table below, the "other" category includes codes that were not closely related to pre-determined codes and in which only one PSTs' statement was identified with this code. In some cases, the results from one of the items helped to provide greater insight about a research question than the other item, so for these research questions, only data for one item has been shared.

Question 1: What explanations do the PSTs use as they solve linear equations?

The verbal explanations offered by PSTs while solving Item 1 varied greatly. As shown in Table 1, 115 PST statements were coded as an explanation, strategy or model. Of these codes, 79 were coded in response to Item 1, and 36 were coded in response to Item 2. These codes were not merely PST descriptions of what they were doing, but rather instances in which the PSTs were providing reasons for operating on the equation in particular ways or providing potential connections that might lead to greater student understanding or awareness.

Table 1

Frequency of Explanation Codes and Number of PSTs Responding

Explanation of Code	Frequency of Code (Number of PSTs)
Order of Operations	15 (6)
Parentheses mean to Distribute and/or Distribution First	7 (6)
Use of Metaphors	12 (7)
Doing the Same to Both Sides	12 (5)
Canceling/Using Inverse Operation	8 (3)
Side Suggestion	6 (5)
Other	15 (6)

The PSTs commonly cited order of operations as a justification for their work. Six PSTs reasoned that they began the problem by distributing the seven due to the first step in the order of operations, namely "parentheses." Adam's statement provides an example of this type of reasoning:

> Okay. Well, I think I would remind them of the order of operations, I think PEMDAS, where I student teach, there's a, on the wall there's a poster that has the pyramid of operations, and I think it's PEMDAS, and so I'd remind them of that. I would ask them, you know, are there any parentheses? So, the student would see the parentheses, and I would ask them to distribute, so we'd wind up with 5x plus 9 equals 2 plus and then you distribute the 7 into the x minus 3, being careful to, keeping that a negative.

Other PSTs also mentioned that either they knew that the parentheses indicate either multiplication or distribution, or that they should begin the solution by distributing. The analyses of the written work for Item 1 showed that all twelve PSTs began by multiplying 7 and (x – 3) to solve the equation. Additionally, seven PSTs modeled the Distributive Property by drawing arrows from the 7 to the x and 3, as shown in Figure 4.

$$5x + 9 = 2 + 7(x - 3)$$

Figure 4.

PST models Distributive Property with arrows.

Seven PSTs used physical metaphors to help describe the process of solving a linear equation. The most common metaphor used was the idea of "balancing" which was mentioned by more than half of the PSTs before the pan balance was introduced in Item 2. For example, in responding to Item 1, Belinda stated:

> I would probably reference like a balance. When I have my actual class, I want to have an actual balance, and show them like if I have so many marbles on one side, if I take two off is it balanced anymore?

One PST explained the idea of balancing using a "seesaw," and another PST compared the process of collecting like terms to sorting laundry.

Twelve coded statements, made by five PSTs, explicitly indicated that they were performing the same operation on both sides of the equation, as Faith explained:

> Since the equal sign is in the middle, they would have to do the same thing to both sides so make them explain that. I've seen teachers put a dash line down the middle so anything you do to this side, you have to make sure you do it to this side, if you cross that line.

The PSTs also used inverse operations with a few explicitly using "inverse" and others, like Gabe, using "cancel" when referring to multiplying by a multiplicative inverse. Gabe explained how he solved the equation by multiplying by a reciprocal:

> And then finally you want to get rid of the 2 on the x, so you can divide by 2 or multiply by the reciprocal, whichever one you're comfortable with, multiplying by the reciprocal is just times ½ to both sides, or divide by 2, and it cancels out to be equal to 14.

Five PSTs made specific suggestions about which side of the equal sign the variable should be on, however their reasoning for this convention varied. Some PSTs solved the equation so the coefficient of the variable was positive, while others arranged the equation so the variable was on the left side. Emma explained her reasoning:

> I always learned the x's on this [left] side I don't know if it's always the correct way because you get a lot of negatives and such, but I do that. I know I'm comfortable writing different, like, more than one step on one line, but then again I have to assess what my students are comfortable with.

In their written work, nine of the twelve PSTs wrote their final solution as "$x = 14$," although five of the nine PSTs initially solved the problem by placing the variable on the right.

Question 2: What language do the PSTs use as they engage in solving linear equations?

Language plays an important role in how individuals come to understand mathematical ideas. As such, it is important that teachers use language precisely and consistently. The data set resulted in 686 individual coded statements relating to the PSTs' use of informal language. We did not code words that are commonly used in mathematics, such as "equation," "variable," and "constant;" rather, by informal language, we refer to everyday language that is used in the mathematical explanations of the PSTs. Of these 686 coded statements, 411 were statements PSTs made in response to Item 1 and the remaining 275 were statements in response to Item 2. The most common informal language was reference to "side," which was used to describe the two equivalent expressions that are linked by an equal sign in an equation. The use of "side" accounted for 142 codes for Item 1 and 69 codes for Item 2. All twelve PSTs used the term "side." What was most striking was the diversity of ways in which "side" was used and the various prepositional phrases that PSTs used with this term, such as "on one side," "on the right/left side," "to one side," "to both sides," and "from both sides." As illustrated in Faith's dialogue provided in the previous section, many PSTs discussed using the laws of equality that state that if two expressions are equal then the same real number can be added, subtracted, multiplied or divided (except zero) to both expressions without changing the equivalence of the expressions. We saw that many PSTs used "sides" to refer to the two equivalent expressions. However, the language students sometimes used interfered with the notion of operating on both sides. For example, Emma discussed this process as "moving":

> Again they can pick whether they want to move the x's, which way they want to move the x's and which way they want to move the constants. I'd probably go ahead and move the 5.

In some cases, the PSTs used this terminology in different ways when solving the equation. For example, during Kassidy's explanation of Item 1a she said, "So we can subtract 7x over and subtract 9 over." Later in the interview, Kassidy described the types of difficulties students might face in solving this problem:

> But this distribution of the 7 and then pulling the x over and then moving the 9 over could be kind of tricky if they weren't exactly sure because that [the entire equation] can look like a mess and that [left side] is equal to that [right side] and not really seeing it exactly.

Kassidy described her steps in the process of solving in three different ways – "subtracting over," "pulling over," and "moving over." Kassidy's frequent use of the word "over" may imply that she is seeing a shift across the equal sign, which is different from Faith's description above of operating on both expressions of the equation. Kassidy's conceptions of "moving" quantities "over" may have caused uncertainty when working with the pan balance, such as when Kassidy began to work on Item 2a:

> If B represents the weight of each box pictured below in ounces. The three B's, and then plus the four is equal to ten [pointing to the pictures]. So then you could say that you moved, you could move these four over [pointing to the trapezoids on the 3B +4 side of the pan balance] and then you would know how much three B of them were…Hold on. I'm just, yeah, this is not how I would go about teaching it. So it's taking, so you would take one off here [pointing to a trapezoid on the 3B +4 side of the pan balance], huh, need to think about this.

Although Kassidy eventually realized that using the pan balance required more than simply "moving" items, her initial difficulty may provide evidence that she had developed some conceptual meaning to "moving over" in the context of solving equations.

Several related words and phrases were used by PSTs to explain a similar idea, including "removes," "goes away," "gets rid of," "cancels," "drops off," "reduces," and "clears [a fraction]." Use of each of these words/phrases suggested that a symbol or set of symbols could be eliminated from the equation. However, some words, particularly "cancel," appear to be used in several different ways. For example, consider the ways the following four PSTs used "cancel":

> Gabe: And then finally you want to get rid of the 2 on the x, so you can divide by 2 or multiply by the reciprocal, whichever one you're comfortable with, multiplying by the reciprocal is just times ½ to both sides, or divide by 2, and it **cancels** out to be, equal to 14. (See Figure 5.)

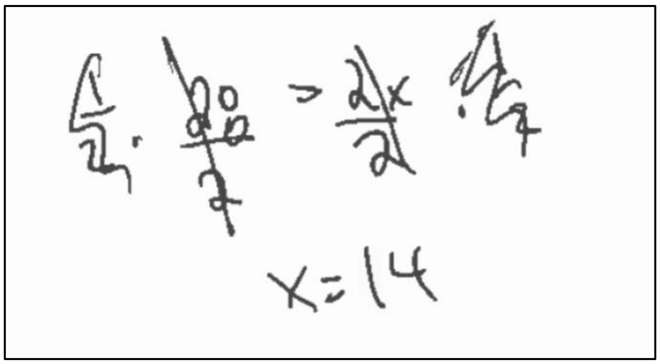

Figure 5.
Gabe solves 28 = 2*x*.

> Isaac: If we wanted to change that 0, another way of saying plus 0 is plus 19, minus 19. And therefore, 9 is equal to 2x minus 19 plus 19 minus 19. And it just so happens that if we then move that 19 over, so plus 19, plus 19, we are left with 28 is equal to 2x, and these will **cancel**. (See Figure 6.)

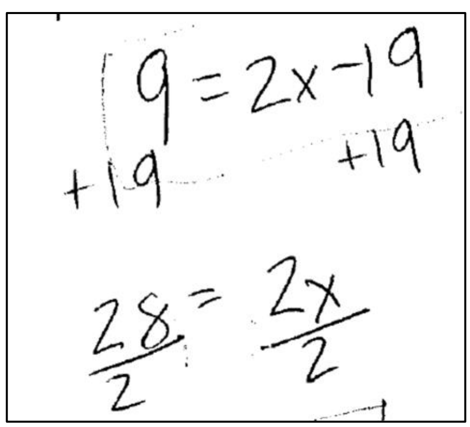

Figure 6.
Isaac solves 9 = 2*x* -19.

Jackson: Alright, so if a student asks you why you can do the same thing to both sides, um what I would tell them is, it's just pretty much, you're kind of almost **canceling** or reducing both sides as in what you do to one side you must do to the other side.

Emma: Cause a lot of students will go through the process of doing problems like this. Like oh, two negatives they go away. But I don't know if they understand what they're actually doing if they multiply on one side, Cause you're getting the negatives **canceling**. (See Figure 7.)

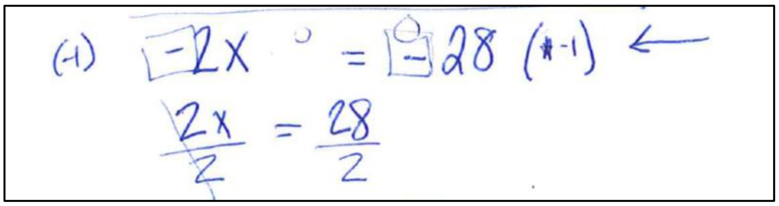

Figure 7.
Emma solves -2x = -28.

Based on their word choice and the explanations given, we can conjecture about how the use of "cancel" is related to the mathematical content. Gabe used "cancel" to indicate the simplification of the fraction $\frac{28}{2}$. Isaac used "cancel" to suggest that adding 19 and subtracting 19 are inverses and can be equivalent to adding zero. In response to Item 1b, Jackson used "canceling" to refer to the process of using an inverse to modify an expression without changing its value. Finally, Emma used the word "canceling" to refer to simplifying the opposite of a negative number. Each of the PSTs used the phrase within the context in which they were speaking that made sense to them individually, yet there was no universal way in which the four students used the term.

Question 3: What benefits and limitations do the PSTs identify about algebraic models used in solving linear equations, such as the pan balance?

Fifty-one PST statements in Item 2 were coded as benefits of the pan balance model, and 52 were coded as limitations. Summaries of these coded statements are presented in Tables 2 and 3.

Table 2

Frequency of Benefits of Pan Balance Statements and Number of PSTs Responding

Benefit Code	Frequency of Code (Number of PSTs)
Visual	22 (10)
Physical/Concrete/Real-life	10 (7)
Builds Connections to Concepts	9 (5)
Other	10 (5)

The majority (10 of 12) of the PSTs cited that one of the benefits of the pan balance is that it provides a visual representation of equation solving. Jackson explained how he understood this benefit:

> So this is a good way to show a visual of finding an unknown weight or anything really. So students can kind of visually see it, cause a lot of students are visual learners and they just can't see numbers and letters and addition, subtraction, multiplication signs, division signs and can solve it.

Seven of the twelve PSTs in the study stated a benefit of the pan balance was that it is a physical object, is a concrete manipulative, or provides a real-life connection. Although this may be true, the PSTs may be making an assumption that the pan balance will react by becoming unbalanced, as Gabe described:

> Well, if I were to use the pan balance I would take away the four ounces on one side, on this side (circling the four ounces on the left side of the equation) and demonstrate what that does to it, it makes the drop in favor of the other side so they're no longer equal. And then ask the students what I should do to the other side to make them equal again.

Gabe, like several other PSTs talked about the pan balance as if it was a physical object that would react when he acted on it. However, the pan balance in Item 2 was not a physical object, but a paper-based model. The picture of the pan balance would not portray a physical action and would not respond to the students' actions in a way that changes the balance; therefore, this benefit could not be realized without a physical representation.

While the paper-based model would not provide the visual of the balancing effect, several PSTs identified ways that the representation could deepen students' conceptual understanding, specifically the students' abilities to know that we operate on both sides in order to maintain equality. For example, Adam described how a pan balance can be used:

> It's more of a conceptual thing that helps them understand what's happening. Why they can subtract the same thing from both sides.

As shown in Table 3, five PSTs mentioned that one of the limitations of the pan balance model is that it is not a good representation for all equations. The PSTs cited the complexity of modeling equations involving decimals, parentheses, and multiple variables.

Furthermore, many of the same participants that cited the visual benefits of the pan balance also encountered difficulties when modeling how to multiply or divide both sides of the equation. While solving Item 2 using the pan balance, Belinda removed four squares from each side of her model, wrote "3B=6" and stated:

> Um, I don't really know what I would do after that, the only way it would work is if these are actually weighted. [points to 3 squares on left] To where these equal, cause B would be 2. So these [points to squares] actually weighed the same as 2 of these. [points to trapezoids] So, if I took one off, how many would I have to take off

here? And so then, and hopefully, then the balance would actually move, cause I only want one of these [squares], so we can take one off, how many can we take off here, we take off two. And same thing, how many do we, two. And then we have one and two left. So, one of these [squares] equals two of these [trapezoids].

Table 3

Frequency of Limitations of Pan Balance Statements and Number of PSTs Responding

Limitation Code	Frequency of Code (Number of PSTs)
Not a Good Model for All Equations	10 (5)
Difficult to Model Multiplication/Division	10 (5)
Cumbersome/Time-consuming	9 (5)
Students Will Not Learn Symbol Manipulation	6 (4)
Other	17 (6)

Furthermore, many of the same participants that cited the visual benefits of the pan balance also encountered difficulties when modeling how to multiply or divide both sides of the equation. While solving Item 2 using the pan balance, Belinda removed four squares from each side of her model, wrote "3B=6" and stated:

> Um, I don't really know what I would do after that, the only way it would work is if these are actually weighted. [points to 3 squares on left] To where these equal, cause B would be 2. So these [points to squares] actually weighed the same as 2 of these. [points to trapezoids] So, if I took one off, how many would I have to take off here? And so then, and hopefully, then the balance would actually move, cause I only want one of these [squares], so we can take one off, how many can we take off here, we take off two. And same thing, how many do we, two. And then we have one and two left. So, one of these [squares] equals two of these [trapezoids].

As Belinda's dialogue illustrates, she was uncertain how to model the division by 3 when using the pan balance. As a result, Belinda also identified this as a limitation to the pan balance model as an instructional tool, saying:

> Mostly, once I get to this step, [points to 3B=6], dividing and multiplying, I'm not sure how I would do that on the balance. I'm sure there's probably a way, I'm just have no idea how to do it.

In nine coded statements, five PSTs suggested a major limitation of the pan balance model is that it is a cumbersome and inefficient method for solving equations. Adam expressed his concerns about how the pan balance should be used:

> We don't want them to take an equation that they get on a test or quiz and draw this diagram of a balancing scale. So, I think a limitation would be that we're introducing it to help them understand what's happening, but we're not introducing it to show them a way to answer the question. I don't think we're introducing the balance as a computational tool that they should use on homework, tests, and quizzes.

Other PSTs echoed Adam's comments that the pan balance would not be appropriate for assessments, or they explained that the pan balance would take a lot of class time. Kassidy mentioned that she would only use the pan balance during demonstrations, rather than have students solve problems this way.

Four PSTs mentioned that the emphasis on the pan balance model might limit students' ability to transfer the knowledge to its symbolic form. Isaac expressed his concern:

> I think the big thing about solving equations is just knowing how to do it. And I think that's a limitation that would arise as if, people might get caught up, students might get caught up on how exactly, for example, on how to write this, the equation that represents this. And that's not what, you know, that's not what problem solving should be or solving equations should be about. It should not be about how to write the equation. It's more about, you know, getting one side by itself. And I think that's the big limitation is problems could arise. Like I said, the critical thinking, you know, they can't understand how to write it. Then they're not going to solve it obviously. And that's what it should be about.

These PSTs suggested that while the pan balance model may be helpful in conceptualizing the equal sign, it may also hinder students' ability to solve equations symbolically.

Discussion

From a research perspective, the study expands the prior work of mathematics educators, who have described teacher knowledge at the elementary level, by using the MKT framework to investigate the algebraic reasoning of secondary teachers. Teachers make a plethora of choices each day that impact students' learning in both explicit and implicit ways, as students may develop conceptual differences in how they understand equation solving. Whereas many PSTs focused on the equation solving (CCK) aspect of the questions, they also utilized their knowledge of the other domains, particularly KCS and KCT. Even before being asked about the balance model, many PSTs mentioned this pedagogical strategy as a way for students to better understand how to solve a linear equation. When continuing this discussion, they were successfully able to determine the strengths of such a model, and some of its weaknesses. Some PSTs considered the differences between a physical representation and a paper-based model, and explored the potential difficulty that students may have with translating this representation into a symbolic one. Some PSTs considered challenges that their students might face in other questions (KCS) and were flexible in giving more than one way to solve a problem (SCK), for example, in the case of inverses and reciprocals. They consistently used terms that they thought would help their students to understand the concepts and showed concern when giving explanations that their students should feel comfortable with the material. Although we have attempted to identify the types of MKT that we feel were best addressed by particular questions and the PSTs' responses, we contend that this classification was not always easy and we experienced some of the same concerns that Speer et al. (2005) described in trying to use the MKT framework with secondary PSTs.

Based on the evidence presented in our study, the types of explanations, language, and models used by PSTs when solving linear equations vary greatly. In some cases these differences may impact how K-12 students understand mathematics and communicate their own algebraic thinking. In regards to issues of language and representation, teachers make choices about when to use formal mathematical language and when more informal, colloquial language is appropriate. For example, how might a student perceive the differences between using the phrase "adding to both sides" and "moving to the other side?" We hypothesize that the PSTs in this study may not realize that the language and processes they have grown accustomed to using themselves may not be mathematically grounded and may interfere with the ways in which their students come to understand the mathematical properties used in solving linear equations. Furthermore, particular words, such as "cancel," are used to convey different mathematical processes. Are teachers attentive to their use of such terms and do they understand the implications of their use? Recent work by Herbel-Eisenmann and Otten (2011) indicated that although capable participants in discourse may be able to move between meanings relatively easily, using words in different ways may be problematic because it can create confusion for students, particularly if these differences are not made explicit.

The results from our study suggest, at least in part, the types of MKT that PSTs need opportunities to develop in their undergraduate education. For example, PSTs need time to unpack and critically examine the reasoning they use when solving linear equations and contrast it to other mathematical processes, such as simplifying expressions using order of operations. They also need time to engage in mathematical practices such as attending to precision in their language and notation. We claim that such fluency in mathematical communication is a type of specialized content knowledge that Ball et al. (2008) described, and is critical for the profession of teaching.

Recent calls from the Conference Board of the Mathematical Sciences (2012) recommended that secondary mathematics teacher education programs require at least three courses that focus on secondary mathematical content from an advanced perspective. However, recent results have shown that few teacher education programs require such courses (Author, 2014). We not only agree that PSTs would benefit from such courses, but based on the data presented in this study, we additionally recommend that these courses address precision of language and notation needed to teach mathematics effectively. It is unclear to what extent mathematical language and word choice has been a part of teacher education programs or professional development experiences for teachers. Our study indicates that teachers need opportunities to reflect on their own mathematical reasoning and examine their language and notation in order to understand how their own ways of thinking and communicating may impact the learning of their students.

Our future plans are to extend these analyses beyond linear equations as we suspect that examining PSTs' interactions with other mathematics content will not only provide greater insight into their mathematical knowledge for teaching, but may also provide insight into how their reasoning with and understanding of linear equations impacts their work with their future students. For example, examining the solving of rational equations may help us see how the explanations and language use related to "cancelling" may influence how PSTs solve and teach this content. The results of our research have the potential to influence the preparation of secondary mathematics teachers as we learn more about what PSTs know and are able to do in relation to equation solving.

Attribution. The authors thank Aladar Horvath, Sarah Kasten, and Frances Keating for assistance collecting and analyzing data for this study.

References

Andrews, P., & Sayers, J. (2012). Teaching linear equations: Case studies from Finland, Flanders and Hungary. *The Journal of Mathematical Behavior*, *31*(4), 476-488.

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics*, *14*(3), 24-35.

Asquith, P., Stephens, A.C., Knuth, E.J., & Alibali, M.W. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: Equal sign and variable. *Mathematical Thinking and Learning*, *9*, 249-272.

Author. (2014).

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*, 389–407.

Chazan, D., Yerushalmy, M., & Leikin, R. (2008). An analytic conception of equation and teachers' views of school algebra. *The Journal of Mathematical Behavior*, *27*(2), 87-100.

Conference Board of Mathematical Sciences (CBMS). (2012). *The mathematical education of Teachers II* (*Issues in Mathematics Education, Vol. 17*). Providence, RI: American Mathematical Society.

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics, 2*9, 1-20.

Floden, R., Ferrini-Mundy, J., Senk, S., Reckase, M.D., & McCrory, R. (2010). *Survey of Knowledge for Teaching Algebra, Form 2*.

Glaser, B. G. & Strauss, A. L. (1967). The discovery of grounded theory: Strategies for qualitative research. New Brunswick, CT: Aldine Transaction.

Herbel-Eisenmann, B. A. & Otten, S. (2011). Mapping mathematics in classroom discourse. *Journal for Research in Mathematics Education, 42*, 451-485.

Huntley, M. A., Marcus, R., Kahan, J., & Miller, J. L. (2007). Investigating high-school students' reasoning strategies when they solve linear equations. *The Journal of Mathematical Behavior*, *26*(2), 115-139.

Kilpatrick, J., Blume, G., Heid, M. K., Wilson, J., Wilson, P., & Zbiek, R. M. (2010). Mathematical understanding for secondary teaching: A framework. In M. K. Heid, P. S. Wilson, & G. W. Blume (Eds.), *Mathematical understanding for secondary teachers: A framework and classroom-based situations* (pp. 9-30). Charlotte, NC: Information Age Publishing.

McCrory, R., Floden, R., Ferrini-Mundy, J., Reckase, M.D., & Senk, S.L. (2012). Knowledge of algebra for teaching: A framework of knowledge and practices. *Journal for Research in Mathematics Education, 43*, 584-615.

National Council of Teachers Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: Authors.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. New York: Routledge & Kegan Paul.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Speer, N. M., King, K. D., & Howell, H. (2015). Definitions of mathematical knowledge for teaching: Using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education*, *18*, 105-122.

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction*, *18*(6), 565-579.

Strauss, A., & Corbin, J. (1998). Basics of qualitative research: Techniques and procedures for developing grounded theory. *Thousand Oaks, California: Sage Publications*.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*, 313-340.

Stueben, M. A., & Torbert, S. M. (2006). Advice for solving equations. *Mathematics Teacher*, *99*(8), 538-542.