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Pre-Service Elementary Teachers' Knowledge of Number Properties and Patterns in the Context of early Algebra

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Abstract

The purpose of this study was twofold: 1) investigate preservice teachers' knowledge of the associative property for addition and multiplication and the distributive; 2) investigate preservice teachers' knowledge of explicit and recursive representation of numerical pattern. In the Common-Core State Standard for Mathematics (CCSSM), fifth-grade students are expected to recognize the usefulness of number properties in simplifying and interpreting numerical expressions. In the CCSSM, fifth-grade students are also expected to learn to analyze patterns and relationships and to generate a numerical pattern given a rule. So, this study was conducted to examine preservice elementary teachers' knowledge of the content they are eventually expected to teach.

Keywords**:** Preservice Elementary Teachers, Teacher Preparation, Early Algebra, Number Properties, Structural Properties, Structure, Pattern, Patterning, Modeling, Algebra

Introduction

Current standards and recommendations for elementary-school mathematics call for the inclusion of *early algebra* in the elementary-school curriculum. Early algebra refers to an elevated focus on algebraic thinking and algebraic concepts that are accessible to elementary-school students (Carraher, Schliemann, & Schwartz, 2008). It is important to note that early algebra does not mean bringing traditional algebra, taught at the secondary-school level, to the elementary-school grades (Carraher et al., 2008). Advocates of early algebra, such as Maria Blanton and her colleagues, have contended that school algebra should be taken as a K–12 content strand (Blanton & Kaput, 2005, 2011; Blanton et al., 2015). These advocates contend that taking school algebra as a K–12 learning experience allows students to "have long-term, sustained algebra experiences in school mathematics, beginning in the elementary grades" (Blanton et al., 2015, p. 40).

The inclusion of early algebra in the school curriculum is evident in the recent standards put forward by the National Council of Teachers of Mathematics (NCTM) (2000, 2006) and in the *Common Core State Standards for Mathematics* (hereafter "CCSSM") (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010—hereafter cited as "CCSSM, 2010"). The stated goal for school algebra in the NCTM (2000) Standards was for "instructional programs from pre-kindergarten through grade 12 [to] enable all students to—

- understand patterns, relations, and functions;
- represent and analyze mathematical situations and structures using algebraic symbols;

- use mathematical models to represent and understand quantitative relationships;
- analyze change in various contexts" (p. 37).

This goal is consistent with the perspective presented by Blanton et al. (2015) to formulate algebra education as a K–12 learning experience. Likewise, the CCSSM (2010) document included specified standards for school algebra in all grades. The inclusion of early algebra learning expectations is evident in the content strand labeled as *Operations and Algebraic Thinking* for grades K–5. Algebra learning expectations for middle-school students are specified as part of the *Expressions and Equations* content strand for grades 6–8. The CCSSM (2010) included a general standard for high school algebra.

Research Focus

The purpose of this study was to examine preservice elementary teachers' readiness to teach concepts related to early algebra. The focus was on preservice teachers' knowledge of content related to the *modeling* and *structural* aspects of early algebra. These two aspects of school algebra are evident in the NCTM (2000) Standards and in the CCSSM (2010) document. The NCTM (2000) Standard for algebra across all grades includes the requirement for instructional programs to ensure that all students were taught to reason mathematically in their endeavor to understand patterns, relations, functions, and structure in mathematics. CCSSM (2010) stated mathematical practices include: modeling with mathematics, looking for and using structure, and looking for and expressing regularity in repeated reasoning.

In this study, I investigated preservice elementary teachers' knowledge of structural properties, such as the associative properties for addition and multiplication and the distributive property for multiplication over addition, and their knowledge of explicit and recursive representations of numerical patterns. I contend that the inclusion of early algebra is the school curriculum without proper teacher preparation to teach early algebra is futile. Blanton et al. (2015) contended that the adoption of CCSSM (2010) in the absence of proper algebra instruction that meets the current standards would only leave students vulnerable to failure.

Number Properties in the Common Core Standards

Preparers of the common core standards for mathematics considered number properties so important that they included a glossary that contained a list of the pertinent number properties in school mathematics (see CCSSM, 2010, p. 90). The following CCSSM content standards for grades 1 and 3 contain language relating to the relevant number properties that are expected to be learned in the elementary-school grades.

CCSS.MATH.CONTENT.1.OA.B.3

> Apply properties of operations as strategies to add and subtract. Examples: If $8 + 3 = 11$ is known, then $3 + 8 = 11$ is also known (commutative property of addition). To add $2 + 6 + 4$, the second two numbers can be added to make a ten, so $2 + 6 + 4 = 2 + 10 = 12$ (associative property of addition). (CCSSM, 2010, p. 15)

CCSS.MATH.CONTENT.3.OA.B.5

> Apply properties of operations as strategies to multiply and divide. Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known (commutative property of multiplication). $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$ (associative property of multiplication). Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can

> find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56 (distributive property). (CCSSM, 2010, p. 23)

These are not the only standards related to number properties that were listed in the CCSSM (2010) for grades 1–5. This is a sample of what is expected to be learned about structural properties in the elementary-school grades. By the fifth grade, students are expected recognize the usefulness of structural properties in simplifying and interpreting numerical expressions without having to evaluate them.

Modeling in the Common Core Standards

The modeling aspect of school algebra has received the most attention in mathematics education research when compared to the structural aspect in the literature (Kanbir, Clements, & Ellerton, 2017). There are many researchers in the mathematics education community who have put forward the idea that algebra ought to be, and can be, successfully introduced in the elementary-school grades through the modeling and functional thinking approach (see, Blanton et al., 2015; Blanton & Kaput, 2011; Cai & Knuth, 2011; Kaput, 1998; Radford, 2006, 2011). The modeling aspect is evident across all grades in the CCSSM (2010) document. In the elementary-school curriculum, grade 3 students are expected to learn to "solve problems involving the four operations and identify and explain patterns in arithmetic" (CCSSM, 2010, p. 23). Grade 4 students are expected to learn to generate and analyze patterns and to describe them both explicitly and recursively. Grade 5 students are expected to learn to analyze patterns and relationships and to generate a numerical pattern given a rule.

CCSS.MATH.CONTENT.4.OA.C.5

> Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way. (CCSSM, 2010, p. 29)

CCSS.MATH.CONTENT.5.OA.B.3

> Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns and graph the ordered pairs on a coordinate plane. For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so. (CCSSM, 2010, p. 35)

These standards are presented to offer some insight of how the CCSSM (2010) treats the modeling aspect of school algebra. The standards presented above are those related to the explicit and recursive representations of numerical patterns.

The Basis for the Present Study

It should be evident from the discussion and presentation of samples from the CCSSM (2010) content standards for elementary-school grades that both the structure and modeling treatments of algebra, and of mathematics in general, are emphasized in the common core standards for mathematics. Furthermore, early algebra researchers, such as Kaput and colleagues (see Kaput, 2008; Kaput, Carraher, & Blanton, 2008) have defined school algebra as the study of structures, including those stemming from arithmetic, and also as the study of patterns, functions, and relations. It is from this perspective that I frame the present study. This study is concerned with the early algebra content that preservice elementary teachers are expected to teach upon completion of their teacher education programs.

In this study, I restricted my investigation to preservice teachers' knowledge of *structure* and *patterning* in mathematics. The following definitions for structure and patterning were used in this study. Structure refers to the use of number properties to solve and simplify problems in algebra. Patterning refers to expression of numerical patterns explicitly and recursively, and the ability to generate patterns given their *explicit* and *recursive* rules. Explicit representation of patterns allows one to find the value of any term in the pattern given the rule. Recursive representations are those rules that allow one to find the n^{th} term of the pattern when the previous term is already known, in addition to the first term.

These definitions for structure and patterning are comparable to those used by Kanbir, Clements, and Ellerton (2017). Kanbir and colleagues investigated seventh-grade students' knowledge and understandings of structural properties of real numbers and their ability to recognize and describe patterns explicitly and recursively. The present study is an investigation of whether preservice elementary teachers, in their final content-related mathematics methods course, can do the same. I investigated whether the participating preservice teachers had strong knowledge and understanding of structural properties, and whether they could represent number patterns explicitly and recursively. As previously shown, elementary-school students are expected to learn about number properties as well as explicit and recursive expressions of numerical patterns by the end of the fifth grade according to the Standards document prepared by CCSSM (2010). The theme of patterns, relations, functions, and structure in mathematics is evident across all grades in the NCTM (2000) Standards. So, if preservice elementary teachers are to be effective teachers, they must have thorough knowledge and strong understandings of the mathematics content that they are expected to teach upon completion of their teacher preparation programs.

Related Literature

Despite the revamping of mathematics school curricula to include early algebra, evidence from research indicate that students' difficulties with algebra persist (Kanbir et al., 2017). It is not evident that the kind of algebra learning expected from the authors of the Standards documents (CCSSM, 2010; NCTM, 2000, 2006) is happening in the classrooms (Kanbir et al., 2017). You (2006) attributed students' challenges with school algebra to the kind of mathematics instruction they received in the schools. I contend that the algebra instruction recommended by the NCTM (2000) and in the CCSSM (2010) is not occurring in the elementary and middle-school grades because many beginning teachers are not prepared to teach the mathematics recommended in the Standards documents. Ellerton and Clements (2011), for example, conducted a study investigating prospective middle-school mathematics teachers knowledge of equations and inequalities. They found that many of the preservice teachers seeking a middle-school teacher certification who participated in their study did not have strong understandings of the mathematics they were expected to teach after completing their teacher education programs.

In the case for structural properties, Ding, Li, and Capraro (2013) and Ding (2016) conducted studies in which they investigated preservice teachers' knowledge for teaching the associative property of multiplication. They found that the participating preservice teachers had trouble differentiating between the associative properties and the commutative properties. They also found that most preservice teachers in their studies "were unable to use concrete contexts (e.g., pictorial representations and word problems) to illustrate [the associative property] of multiplication conceptually, particularly due to a fragile understanding of multiplication" (p. 36). In their study, Ding et al. (2013) also found that the textbooks used by preservice teachers did not provide conceptual support for teaching the associative property of multiplication—both at the university and the elementary-school levels.

The studies of Kanbir et al. (2017) and Ma (1999) support the idea that learning algebra is not a unidimensional trait. Kanbir et al. (2017) found that learning algebra from the structural perspective did not support the modeling aspect. Likewise, learning algebra from the modeling perspective did not support the structural aspect. To receive a full and comprehensive treatment of school algebra, students should be afforded opportunities to learn algebra from both the structural and modeling aspect. Ma (1999) contended that Chinese elementary-school teachers had stronger understandings of structure in elementary mathematics when compared to their counterparts in United States. Ma argued that an effective elementary-school teacher needed to attend to strong knowledge of structure in mathematics. Kanbir et al. (2017) proposed that if Ma's (1999) claim were to be accepted, then knowledge of structure in algebra ought to be emphasized and given a significant level of focus in teacher preparation. In my study, I do not engage in the debate of which approach toward school algebra is more effective. I contend that students should be afforded the opportunities to experience algebra through both structure and modeling.

Methods

In this study, I investigated preservice elementary teachers' knowledge of the associative properties for addition and multiplication and the distributive property. I also investigated what preservice elementary teachers (PETs) knew about explicit and recursive representation of number patterns before and after instruction. The following research questions were used to guide this study:

1. Were there statistically significance differences in the overall performance of PETs on the pretest and posttest measures?
2. Were there statistically significance differences in the performance of PETs on the structural components of the pretest and posttest measures?
3. Were there statistically significance differences in the performance of PETs on the patterning components of the pretest and posttest measures?
4. Were PETs able to recognize the usefulness of the associative properties for addition and multiplication, and the distributive property, in simplifying numerical expressions before instruction?
5. What knowledge did the class of PETs exhibit about recursive and explicit representations of number patterns, before and after instruction?

Study Design, Participants, and Procedure. This study featured a pretest-posttest design. The data used in this study were based on performances on pre- and post-teaching algebra tests that were administered to PETs. The PETs who participated in this study were enrolled in a mathematics methods course for preservice elementary teachers at a large public university in U.S. Midwest. This mathematics methods course (hereafter "MFT2") was the final mathematics content-related course that the PETs were required to take in partial fulfilment of their teacher preparation program. In this program, there were two mathematics courses that were required for pre-service teachers, and MFT2 was the second and final course in the sequence.

MFT2 was a semester-long course that was divided into three units: algebra, geometry, and measurement. Each of these units received about the same amount of instructional time. This study is concerned with the algebra unit of MFT2 which lasted for approximately 5 weeks. There were no measures taken to control who was enrolled in MFT2. Given the reality of the university's procedures for course enrollment, PETs could not be randomly selected to join MFT2. Eighteen PETs were enrolled in the course. Seventeen of them gave permission to have their pre- and post-teaching algebra tests used in this study.

The algebra unit of the MFT2 course included additional topics on number properties and patterning that were not part of the regular syllabus. The pre- and post-teaching algebra tests designed by Kanbir et al. (2017) were adopted and used in this study with permission. Kanbir et al. (2017) designed the tests to assess students' knowledge and understanding of the associative properties for addition and multiplication and the distributive property as well as students' knowledge of explicit and recursive representations of number patterns. Both the pretest and posttest had 15 questions. Question 12 had two parts to it, and question 14 had three parts to it. So, there was a total of 18 items on both tests. All 18 items were open-ended questions. PETs were given 1 hour to complete the pretest and posttest when each was administered.

Data Analysis

Responses to the pretest and posttest were coded for correctness as wells as strategy employed. After the responses had been coded for correctness, the pretest and posttest measures were scored, and each participating PET was assigned an overall score for each test. There were 18 items on each of the tests used in this study; Nine of them pertained to the use of structural properties; and the other 9 items pertained to explicit and recursive representation of number patterns. Each participating PET was also assigned a score for structure and for pattern for both the pretest and posttest.

After responses had been coded according to correctness, the overall score and scores related to the both structural and patterning components of both the pretest and posttest were recorded and analyzed quantitatively. Some of the questions, like those shown in Figure 1, were coded for both correctness and strategy. These types of questions were generally expected to be easy for PETs, but I was interested in examining whether strategies PETs used would change after instruction had taken place.

Results

Descriptive statistics of the pretest and posttest data are shown in Table 1. A paired-samples t-test was conducted to test for differences in the performance of PETs on the pretest and posttest measures. The paired-samples t-test was conducted three times to address the first, second, and third research questions. The first paired-samples t-test was conducted to test whether there was a significant difference in the overall performance of PETs on the pretest and posttest. This test was significant, $t(16) = -5.59$, $n = 17$, $p < .001$. There was a statistically significant difference in the overall performance of PETs on the pretest and posttest. On average, pretest scores (M = 11.29, SD = 2.76) were 3.71 points lower than posttest scores (M = 15.00, SD = 2.00).

Pretest	**Posttest**
Q3: Suppose you were asked to calculate the value of 940 + (60 + 403) in your head (without writing anything down or using a calculator). How would you do it, and which property would you be using? Q15: What would be a quick method of finding the value of $64 \times (\frac{1}{32} \times 120)$, *without* using a calculator?	Q3: Suppose you were asked to calculate the value of 920 + (80 + 533) in your head (without writing anything down or using a calculator). How would you do it, and which property would you be using? Q15: What would be a quick method of finding the value of $48 \times (\frac{1}{24} \times 150)$, *without* using a calculator?

Figure 1. Sample of questions that were for both correctness and strategy.

Table 1
Descriptive Statistics of the Pretest and Posttest Data

		Mean	N	SD	SE
Overall	Pre-test	11.29	17	2.76	.668
	Post-test	15.00	17	2.00	.485
Structure	Pre-Test	6.76	17	1.35	.327
	Post-Test	8.18	17	1.02	.246
Pattern	Pre-Test	4.82	17	2.01	.487
	Post-Test	6.82	17	1.55	.376

The second paired-samples t-test was conducted to test whether there was a significant difference in the performance of PETs on the structural components of the pretest and posttest. This test was also significant, t (16) = -3.59, n = 17, p = .002. There was a statistically significant difference in the performance of PETs on the structural components of the tests. On average, scores of the structural component on the pretest (M = 6.76, SD = 1.35) were 1.42 points lower than scores of the same component on the posttest (M = 8.18, SD = 1.02).

The third paired-samples t-test was conducted to test whether there was a significant difference in the performance of PETs on the patterning component of both the pretest and posttest. This test was also significant, t (16) = -3.63, n = 17, p = .002. There was a statistically significant difference in the performance of PETs on the patterning components of the tests. On average, scores of the patterning component on the pretest (M = 4.82, SD = 2.00) were 2 points lower than scores of the same component on the posttest (M = 6.82, SD = 1.55).

The fourth research question was addressed by examining the type of responses that PETs gave to select questions pertaining to structure on both the pretest and posttest. Figure 2 contains samples of questions pertaining to number structure that were in the pretest and posttest. Question 3 (Q3) was assessing PETs ability to recognize that the associative property (AP) for addition would be useful in simplifying the given number expression. Categorization of PETs responses to Q3 on the pretest and posttest is presented in Table 2. On the pretest, some of the respondents recognized the usefulness of associative property for addition. However, most of the respondent relied on the mnemonic PEMDAS (**P**arenthesis, **E**xponents, **M**ultiplication or **D**ivision, and **A**ddition or **S**ubtraction) or some other strategy. On the posttest, nearly all the respondents applied the associative property for addition to simplify the number expression in Q3.

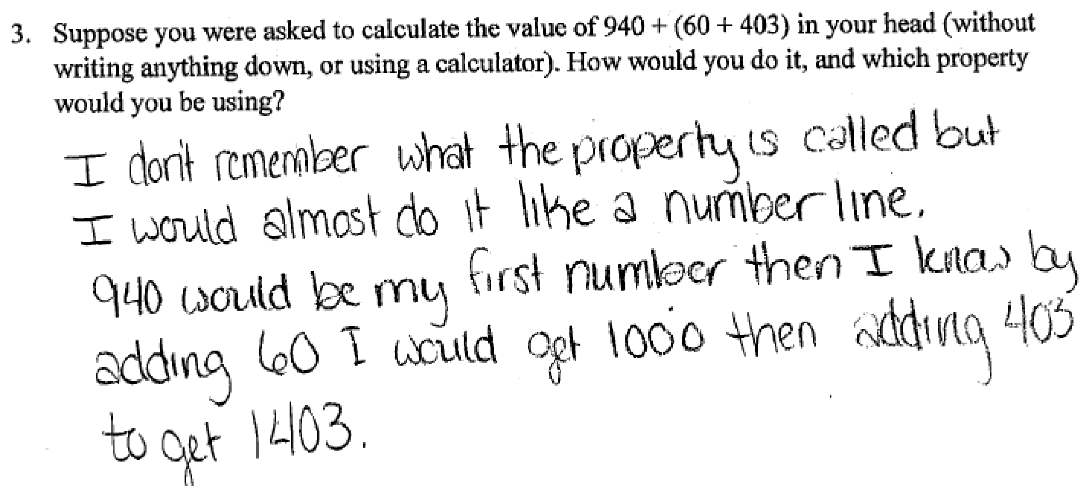
Pretest	**Posttest**
Q3: Suppose you were asked to calculate the value of 940 + (60 + 403) in your head (without writing anything down or using a calculator). How would you do it, and which property would you be using?	Q3: Suppose you were asked to calculate the value of 920 + (80 + 533) in your head (without writing anything down or using a calculator). How would you do it, and which property would you be using?
Q13: What would be a quick method of finding the value of 7 × 97 + 7 × 3 without using a calculator? What is the property which allows you to use that quick method?	Q13: What would be a quick method of finding the value of 8 × 96 + 8 × 4 without using a calculator? What is the property which allows you to use that quick method?
Q15: What would be a quick method of finding the value of $64 \times (\frac{1}{32} \times 120)$, *without* using a calculator?	Q15: What would be a quick method of finding the value of $48 \times (\frac{1}{24} \times 150)$, *without* using a calculator?

Figure 2. Sample of questions pertaining to number structure.

Table 2
Categorization of Responses to Q3

		Frequency	Percent
Pre-Test	Correct, PEMDAS	7	41.2
	Correct, AP	5	29.4
	Correct, Other	3	17.6
	Incorrect, PEMDAS	1	5.9
	Incorrect, Other	1	5.9
	Total	17	100.0
Post-Test	Correct, AP	16	94.1
	Incorrect, AP	1	5.9
	Total	17	100.0

Even though some of the PETs recognized that the associative property for addition could be applied to Q3 on pretest, many of them were not able to name the property before instruction. A sample response to Q3 on the pretest is presented in Figure 3. This observation was not unique to Q3. Similar observations were made on similar question-items. For Question 13 (Q13), PETs were expected to recognize that the distributive property (DP) would be useful in simplifying the given number expression. Table 3 shows the categorization of PETs responses to Q13 on the pretest and posttest. On the pretest, none of the respondents recognized the usefulness of the distributive property. Most of the respondent relied on PEMDAS or some other strategy. On the posttest, nearly all the respondents applied the distributive property to simplify the number expression in Q3.



3. Suppose you were asked to calculate the value of 940 + (60 + 403) in your head (without writing anything down, or using a calculator). How would you do it, and which property would you be using?

I don't remember what the property is called but I would almost do it like a number line. 940 would be my first number then I know by adding 60 I would get 1000 then adding 403 to get 1403.

Figure 3. Sample response to Q3 on pretest.

Table 3

Categorization of Responses to Q13

		Frequency	Percent
Pre-Test	Correct, PEMDAS	4	23.5
	Correct, Other	2	11.8
	Incorrect, PEMDAS	2	11.8
	Incorrect, DP	1	5.9
	Incorrect, Other	8	47.1
	Total	17	100.0
Post-Test	Correct, DP	16	94.1
	Incorrect, DP	1	5.9
	Total	17	100.0

Question 15 (Q15) was assessing PETs ability to recognize that the associative property (AP) for multiplication would be useful in simplifying the given number expression. Table 4 below show categorization of PETs responses to Q15 on the pretest and posttest, respectively. On the pretest, some of the respondents recognized the usefulness of associative property for multiplication. However, most of the respondent relied on PEMDAS or some other strategy. On the posttest, nearly all the respondents applied the associative property for multiplication to simplify the number expression in Q15. On both the pretest and posttest, one of the PETs incorrectly applied the distributive property to Q15.

The fifth research question was addressed by examining the type of responses that PETs gave to questions pertaining to pattern in algebra on both the pretest and posttest. Figure 4 contains samples of questions about patterning that were in the pretest and posttest. A majority of the participating PETs answered Question 4 (Q4) correctly on the pretest. However, many of them had circled the question-item or had placed a question mark or some other indicator around Q4.

Table 4

Categorization of Responses to Q 15

		Frequency	Percent
Pre-Test	Correct, PEMDAS	3	17.6
	Correct, AP	6	35.3
	Incorrect, PEMDAS	5	29.4
	Incorrect, Other	2	11.8
	Incorrect, DP	1	5.9
	Total	17	100.0
Post-Test	Correct, AP	14	82.4
	Incorrect, AP	1	5.9
	Incorrect, Other	1	5.9
	Incorrect, DP	1	5.9
	Total	17	100.0

A sample response to Q4 on the pretest is presented in Figure 5. I hypothesize that many of the PETs had seen this type of problem in their prior algebra experience, but the subscript notation was new to them. So, this new notation may have created confusion about was expected for Q4. On posttest, all responses to Q4 were correct.

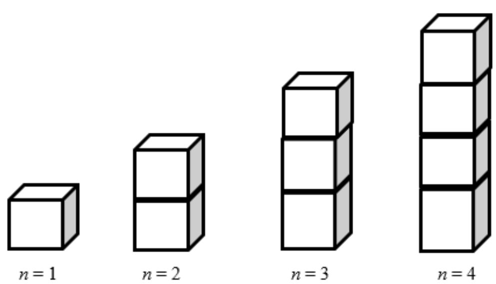
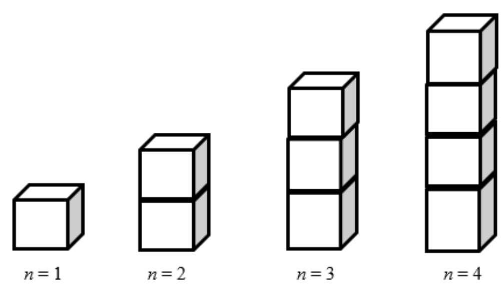
Pretest	**Posttest**
Q4: If $S_n = 101 + 50n$, where n can be any whole number, what is the value of S_4? Q5: A student is creating towers out of unit cubes. Each unit cube, by itself, has 6 square faces, but when two unit cubes are stuck together, one exactly on top of the other, there are only 10 faces in the tower (including the top and the bottom). The first tower has 1 unit cube and 6 faces. The second tower has 2 unit cubes, one on top of the other, and the third tower has 3 unit cubes, etc. We say that the surface area of the first tower is 6 units, of the second tower is 10 units, etc.  What is the surface area of a tower with 50 cubes? Q6: If $T_n = 2n + 5$, what is value of $T_{n+1} – T_n$?	Q4: If $S_n = 102 + 40n$, where n can be any whole number, what is the value of S_5? Q5: A student is creating towers out of unit cubes. Each unit cube, by itself, has 6 square faces, but when two unit cubes are stuck together, one exactly on top of the other, there are only 10 faces in the tower (including the top and the bottom). The first tower has 1 unit cube and 6 faces. The second tower has 2 unit cubes, one on top of the other, and the third tower has 3 unit cubes, etc. We say that the surface area of the first tower is 6 units, of the second tower is 10 units, etc.  What is the surface area of a tower with 50 cubes? Q6: If $T_n = 3n + 4$, what is value of $T_{n+1} – T_n$?

Figure 4. Sample of questions pertaining to number pattern.

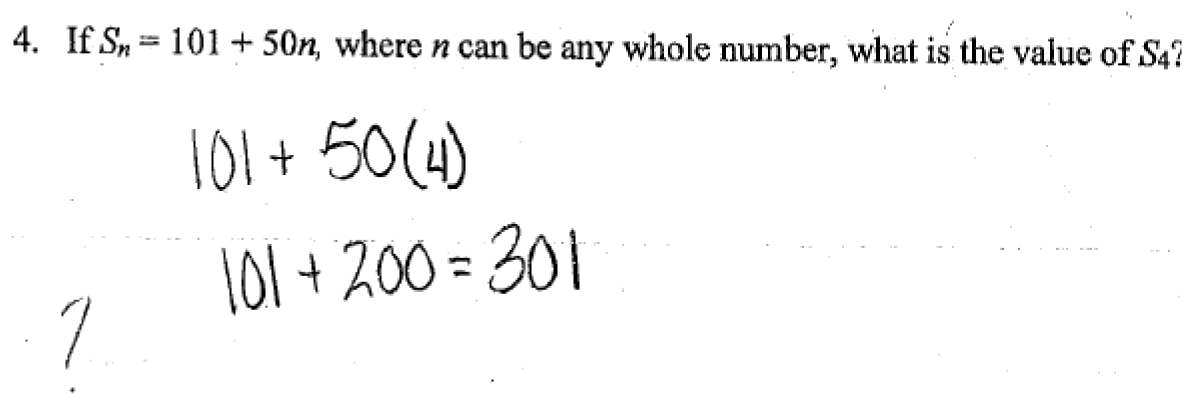


Figure 5. Sample response to Q4 on pretest.

On the pretest, only 41% of the responses to Question 5 (Q5) were correct. There were various reasons PETs incorrectly answered Q5. Some of the PETs tried to answer this question through systematic brute force; they tried to list the number of square faces for all stackings of cubes from a tower of 1 cube to a tower of 50 cubes. Some of the PETs overcounted the number of square faces as more cubes were added to the tower; they included the bottom face that is no longer counted once the cube has been stacked to the tower. Most of those who correctly answered Q5 on the pretest were able to do so because they identified the appropriate rule for generalization. On the posttest, 82% of the responses to Q5 were correct. Question 6 (Q6) was especially challenging to PETs both before and after instruction. All of the responses to Q6 on the pretest were incorrect, and only 29% of the responses were correct on the posttest.

Discussion

The results of this study show that the participating preservice elementary teachers did not have strong understandings of the usefulness of the associative and distributive properties in simplifying numerical expressions at the pretest stage. According to intended curriculum in the CCSSM (2010) documents, fifth-grade students are expected to recognize the usefulness of number properties in simplifying and interpreting numerical expressions. There is reason to suspect that this is not happening in schools. Kanbir et al. (2017) found that seventh-grade students in their study, at the pre-intervention stage, hardly had any knowledge of the associative and distributive properties and were not able to recognize the usefulness of these properties in simplifying expressions. The preservice elementary teachers that were part of this study had completed their elementary and secondary-school education in the Standards era, and yet they did not have thorough understandings of the applicability of number properties prior to instruction. This indicates that the NCTM (2000, 2006) and CCSSM (2010) intended curriculum had not had the effect that was desired by those who prepared the Standards documents.

Similarly, these participating preservice elementary teachers did not have strong knowledge of number pattern prior to instruction. In the CCSSM (2010) document, fifth-grade students are expected to learn to analyze patterns and relationships and to generate a numerical pattern given a rule. Kanbir et al. (2017) found that seventh-grade students in their study, at the pre-intervention stage, scarcely had any knowledge of recursive or explicit representation of patterns. Blanton et al. (2015) wrote that poor preparation of teachers to teach early algebra would only leave young students vulnerable to failure. I contend that the failure had occurred. The remedy is to ensure preservice teachers are afforded adequate support and learning opportunities to meet the current standards for early algebra.

Implications

Because of the design of this study, it would not be appropriate to make far-reaching claims about all preservice elementary teachers. The sample of preservice elementary teachers that participated in this study was not randomly selected. Because of the nature of enrollment at the university where this study was conducted, it was not possible to randomly select those who would enroll in MFT2. However, the results of this study are still useful and meaningful. These results indicate that the class of preservice teachers that was the subject of this investigation did not have thorough understanding, at the pretest stage, of the content they were eventually expected to teach. The MFT2 was helpful to the preservice teachers that took part in the course. Statistical results indicate that there were statistically significant pretest-posttest gains. Similar investigations are recommended with different samples of preservice elementary teachers at different teacher preparation programs. Further investigations with a retention component, like that reported by Kanbir et al. (2017), would be particularly illuminating on whether preservice teachers retain new knowledge several weeks after instruction has taken place.

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