

AN ATYPICAL APPROACH TO IMPROVE TYPICAL ISSUES WITH PRE-SERVICE TEACHERS' GEOMETRIC SHAPE KNOWLEDGE

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Abstract

Although geometric shape content should be mastered by early high school, many pre-service teachers (PSTs), particularly those in elementary and special education, have limited yet deeply engrained shape concepts. This poses challenges for how to teach shape concepts in content courses for PSTs. Thus, this study tested an approach atypical for a content course. PSTs' positive perceptions of this 40-minute activity and the specific ways their content knowledge did and did not change were reported. The results could influence instruction of mathematics content courses for PSTs in powerful ways with minimal time as well as open research opportunities.

Introduction

Concerns about students' learning of geometric concepts around the globe suggest the need to improve the quality of instruction in the early years. Classifying geometric shapes is generally considered a simple task by adults. It has, however, been documented as a challenging area not only for K-12 students but also for adults such as pre-service and in-service teachers, PSTs and ISTs, respectively (Clements & Sarama, 2011; Fujita & Jones, 2007; van der Sandt & Nieuwoudt, 2003). Therefore, there is a pressing need for teacher educators in mathematics education to ensure that PSTs have common content knowledge (CCK) of geometric shapes (Ball, Thames, & Phelps, 2008). Although teachers must also have specialized content knowledge, this article focuses on the primary objective of content courses taught in mathematics departments, that of developing PSTs' CCK (Ball, Thames, & Phelps, 2008; Max & Amstutz, 2019).

Teachers' Level of Reasoning in Geometry

The van Hiele framework is frequently used to examine PSTs' and ISTs' CCK by measuring their level of geometric reasoning. A learner advances through increasingly sophisticated levels of pre-recognition, visual or syncretic, descriptive, informal deduction, formal deduction, and rigor (Clements, Swaminathan, Hannibal, & Sarama, 1999; van Hiele, 1986). To teach effectively, teachers must know considerably more than the content of the grade level they teach (National Mathematics Advisory Panel, 2008). However, 17% of middle school ISTs, 21% of secondary PSTs and 11% of elementary PSTs were at the visual level (Halat, 2008a; Halat, 2008b). Most surprisingly 7% of elementary PSTs were at the precognition level (Halat, 2008b). These results suggest that almost one-fifth of those teaching elementary mathematics were not even at the reasoning level expected of elementary children.

Teachers' Understanding of Shape Categories

Recognizing geometric shapes and exploring their attributes are considered basic concepts, yet many elementary and secondary PST/ISTs struggle with shape knowledge (Halat, 2008a; 2008b; Millsaps, 2013; Pickreign, 2007; Tsamir, Tirosh, Levenson, Barkai, & Tabach, 2015). Often, these difficulties stem from ignoring the properties to primarily identify shapes visually based on prototypical images (Fujita, 2012; Millsaps, 2013; Pytlak & Swoboda, 2017).

In-Service and Pre-Service Teachers' Conceptions of Rectangle and Triangle. A search for teachers' knowledge of triangle yielded just one study with a triangle task. Tsamir et al. (2015) found that about one-sixth (17%) were unable to distinguish between triangles and the 2D distractors that violated at least one property of polygons. Early grade PSTs as well as ISTs have difficulty accepting that every square is a rectangle (Clements & Sarama, 2011; Erdogan & Dur, 2014; Fujita & Jones, 2007; Fujita, 2012, p. 61; Luneta, 2014). When Fujita and Jones (2007) asked 158 elementary PSTs in Scotland, "is a square a rectangle," only 13% agreed. In Turkey only 46% of PSTs identified the rectangles correctly among a variety of quadrilateral images (Erdogan & Dur, 2014). PSTs' definition of rectangle relies on prototypical images, evinced by statements such as "a quadrilateral with two longer and 2 shorter sides" (Pickreign, 2007; Fujita, 2012; Turnuklu, Gundogdu Alayli, & Akkas, 2013). This reflects a partitive definition rather than the more efficient hierarchical definition commonly accepted in mathematics (De Villiers, 1994). The root of such conceptions likely lies in the separate introduction of rectangles and squares (Clements & Sarama, 2011; Nurnberger-Haag, 2017). An analysis of informal sources of shape knowledge in the form of children's books, for example, found that 43% of books that gave children some property of rectangles explicitly told them two sides must be longer than the others (Nurnberger-Haag, 2017).

Conceptual Conflation of 2D and 3D Shapes. Studies have primarily assessed either 2D or 3D shapes. For instance, Tsamir et al. (2015) used only 2D shapes to assess ISTs' conception of circle and a separate assessment only of 3D shapes to assess ISTs' conception of cylinder. Such measures miss the opportunity to capture the conflation of 2D and 3D shapes. However, in a South African study PSTs were asked to draw and name 2D and 3D images (Luneta, 2014). Most (63%) struggled to distinguish dimensionality and drew either a cube or a rectangular prism for square and rectangle respectively. Such results are unsurprising given that a primary source of early shape knowledge (i.e., children's books) conflates 2D and 3D shapes (Nurnberger-Haag, 2017).

Purpose of the study

Typical instructional practices have insufficiently prepared PSTs to teach geometry (van der Sandt & Nieuwoudt, 2003). As a result, there is a need for new short but effective approaches to help PSTs recognize their own incomplete or inaccurate conceptions of shapes in a way that feels less threatening than traditional assessments or instruction and simultaneously improves their CCK. This study reports effects of an activity intended to accomplish these goals in the context of teaching PSTs content within a course in a mathematics department. This report answers the questions: How did PSTs in a mathematics content course perceive their experiences with the atypical activity of evaluating the mathematics in children's books? How, if at all, did PSTs' CCK of geometric shapes and their perception about their CCK change due to this activity?

Method

To investigate whether and in what ways undergraduate PSTs' CCK of geometric shapes would benefit from an instructional activity atypical for a content course in a mathematics department, the following method was followed.

Participants. All sections of the undergraduate PST mathematics content course that addressed geometry experienced the instructional activity. However, only the instructor who had previously implemented this activity chose to have her sections participate in the research study. This participating instructor was a female professor with 26 years experience teaching university mathematics courses, including 19 years teaching content courses for PSTs. The PSTs taking this course were primarily elementary, middle childhood, and special education PSTs. A researcher visited all three sections of the participating instructor's course two months after the activity to collect the anonymized surveys from consenting PSTs ($N=89$) with a response rate of 91.75%.

Instructional Activities. Typical instruction in the mathematics content courses consisted of students correcting homework in class and asking questions, participating in an interactive lecture and doing and discussing additional problems. Thus, the activity reported here was atypical for the content course in this mathematics department. The professor gained experience using this atypical activity with earlier versions of the rubric in three successive years. The data reported here reflects the first research of the activity used with the published open source version of the rating scale (Nurnberger-Haag, 2018). See Nurnberger-Haag, et al. (2020) for a more detailed explanation of this instructional activity. The professor gave the pretest about a week prior to the activity to inform her instruction. Instead of homework problems, PSTs were to read the rating scale. The professor reflected "It was obvious that some had done so but there are always students that don't." The professor provided children's books obtained from the local library and also borrowed some from the first author. The professor asked PSTs in groups of two to three to use the scale to analyze the given book for about 20 minutes. The professor then led a 20-minute discussion in which groups shared their analyses of their book with peers on a document camera.

Data Sources and Analysis

The data sources consisted of a pre-post measure containing survey questions and two content items.

Survey sources and analysis. The post-survey items detailed here consist of one open response item—*What 3 words would you use to describe your experience with this activity?* — followed by five Likert items from strongly agree to strongly disagree: (a) *This activity was worthwhile.* (b) *A mathematical shape should be categorized by how it looks.* (c) *I know the geometric shapes needed to teach the grade levels I plan to teach.* (d) *This activity helped me to realize that I had a misconception about at least one shape.* (e) *Evaluating children's books with the given rubric improved what I know about shapes.* Prior to the activity, PSTs were asked questions b and c. Descriptive statistics and/or a visual display are reported for each item.

Content tasks and analysis. Triangle and rectangle assessment tasks were used to sample PSTs' polygon knowledge. Of the 89 participating PSTs, 81 PSTs submitted their pretest and 86 submitted the posttest. Each item response was dichotomously coded as 1 (correct) or 0 (incorrect). To determine whether there was a significant change in PSTs' conception of triangle and rectangle due to the activity, McNemar's tests were used (non-parametric test often used for dichotomous matched pairs data; McCrum-Gardner, 2008).

Triangle content task. The triangle task consisted of 15 options with the directions: *Cross out each image that is not a triangle.* Eight valid triangles that varied by type and orientation were available and seven distractors. Three-dimensional distractors (photo of an Egyptian pyramid, photo of the back of a cone-shaped party hat on a child's head, illustration of a slice of pizza, illustration of an evergreen tree, and a triangular prism) composed five of the items. The remaining two distractors were a chevron quadrilateral known to be commonly misidentified as a triangle due to its visual appearance a figure with rounded corners and a figure with rounded corners.

Rectangle content task. The rectangle task consisted of 18 options with the directions: *Draw a loop around each numbered image that is a rectangle.* Four valid rectangles were included: an oblong rectangle oriented with the short side parallel to the bottom of the page and a square also horizontally-oriented, as well as an oblong rectangle tilted and a square rotated 45 degrees from horizontal due to the common misconception that this would be a "diamond" but not a square. Three 3D distractors were included (the same triangular prism used in the triangle task and a rectangular prism drawn with word processing software, an illustration of a shipping box, and an illustration of a book). A single item assessed whether rounded corners would be accepted on a horizontally-oriented oblong rectangle. Five parallelograms with no right angles (one of which was a rhombus), two trapezoids with exactly one pair of parallel sides, and one mathematical kite were also included as distractors.

Results

Because of the emphasis in mathematics departments on mathematics content, this report exclusively addresses CCK, although this activity also addresses specialized content knowledge (Nurnberger-Haag et al., 2020). We first report PSTs' perceptions of the activity followed by pre-post content knowledge.

PST Perception of Activity. Parts of speech of PSTs' responses to the question about their experience (*What 3 words would you use to describe your experience with this activity?*) were consolidated to ensure frequencies accurately reflected data patterns (e.g., *interested* and *interesting* changed to *interesting*, etc.). We also compensated for limitations of the software in that *critical thinking* exceeded the character limit, so we truncated this to *criticalthink*. The word cloud in Figure 1 uses font size to visually represent the relative frequencies of the 264 responses the 88 PSTs stated about their experience. Word cloud representations can support readers to distill large sets of text to more easily identify patterns (DePaolo & Wilkinson, 2014). In spite of there being 264 responses, there was such commonality across PST perceptions that the word cloud displays 100% of the words expressed. The top ten words were *interesting*; *fun*; *critical thinking*; *helpful*; *eye-opening*; *surprised*; *informative*; *shocking*; *confused*; and *enlightening*. Note that eight out of these top 10 words reflect positive experiences. Even some negative words, such as *shocking* were about the books being incorrect rather than a negative personal experience while learning. Based on other studies being conducted, PSTs use the word *confused* for a variety of reasons ranging from ideas such as confused as to why books would be published that are wrong to actual confusion with the activity or the content. Further inspection of Figure 1 includes words such as *good*, *knowledgeable*, *insightful*, *exciting*, *enriching*, *pleasant*, *playful*, *positive*, and so forth.

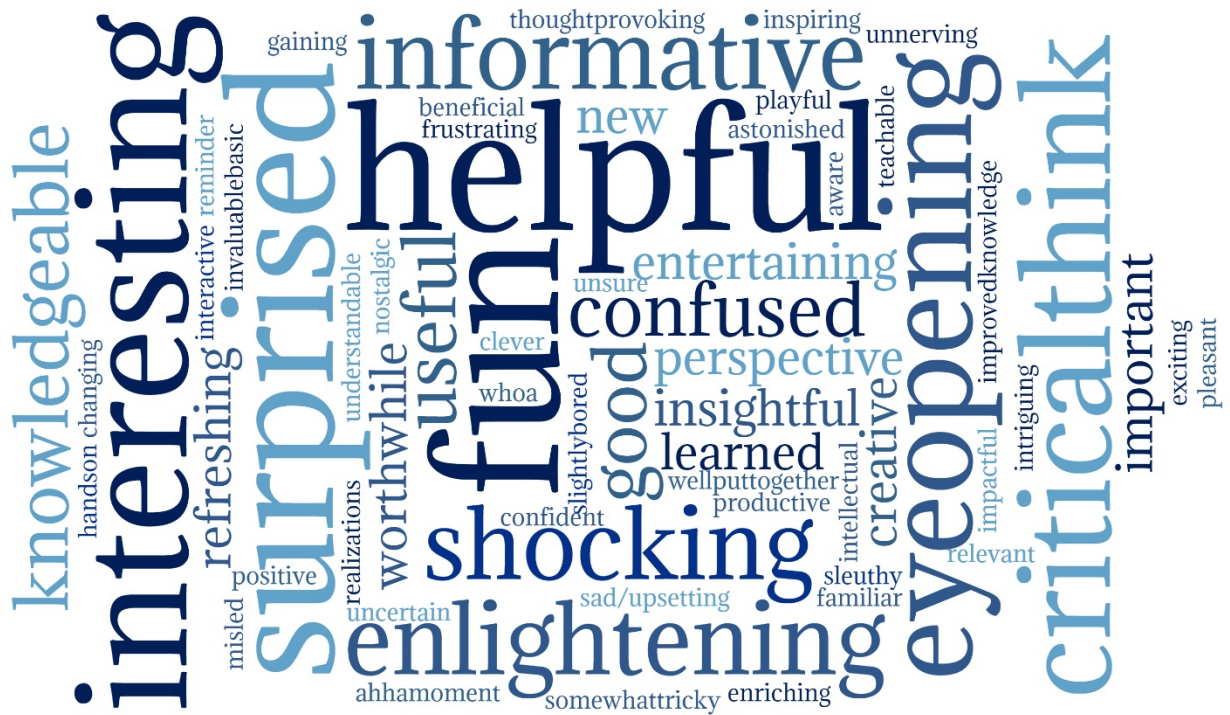


Figure 1

Word Cloud Representation Showing Relative Frequency With Which PSTs Listed These as Words to Describe Their Experience With the Instructional Activity

PSTs overwhelmingly valued this activity as “worthwhile” given that no one disagreed with its value and 97.7% actively agreed. Table 1 displays Likert postsurvey responses about PSTs perceptions of whether the activity was a worthwhile use of time during their mathematics content course and its effect on their CCK. Notice the absence of strongly disagree ratings; few students disagreed with any survey question. With regards to if and how PST content knowledge changed, prior to the activity, at pretest 28% were neutral or acknowledged issues with their own content knowledge. Table 1, however, shows that 82.5% of PSTs acknowledged they had at least one misconception about shapes prior to this activity. Moreover, even some students who did not characterize their knowledge as having a misconception said that the activity improved their knowledge of geometric shapes, such that about 90% agreed it did so. Furthermore, only six PSTs disagreed that they had a misconception and only two disagreed that the activity improved their content knowledge.

Table 1
Postsurvey Responses about Value of Activity for Their Learning

Perception	Agree		Neutral	Disagree	
	Strongly Agree <i>n</i> (%)	Agree <i>n</i> (%)	Somewhat Agree/ Disagree <i>n</i> (%)	Disagree <i>n</i> (%)	Strongly Disagree <i>n</i> (%)
Activity Was Worthwhile	43 (50.0)	41 (47.7)	2 (2.3)	--	--
Activity Revealed I Had At Least One Misconception	31 (36.0)	40 (46.5)	9 (10.5)	6 (7.0)	--
Activity Improved My Content Knowledge	40 (46.5)	37 (43)	7 (8.1)	2 (2.3)	--
After Activity I Know Shapes I Need to Teach	14 (16.3)	53 (61.6)	16 (18.6)	3 (3.5)	--

PST Content Understanding After Instruction. To assess PST content knowledge, we first report the pre-post survey question designed to elicit PSTs’ minimum overall van Hiele reasoning level, that is whether they self-reported that shapes are determined by appearances rather than properties. Then we report the results of the pre-post triangle and rectangle content assessments, which are summarized in Figure 2.

Properties determine shape categories, not visual appearances. The Likert prompt “Shapes should be determined by how they look” reflects reverse coding in that “disagree” reflects higher level reasoning, whereas “agree” reveals a misconception. At pretest only 18.5% correctly disagreed, whereas at posttest almost three times as many disagreed (52.4%). The percent of PSTs who used prototypical thinking at pretest (35.8% agreed) was cut in half after the activity, with only 16.7% still agreeing that the appearance determines the shape name.

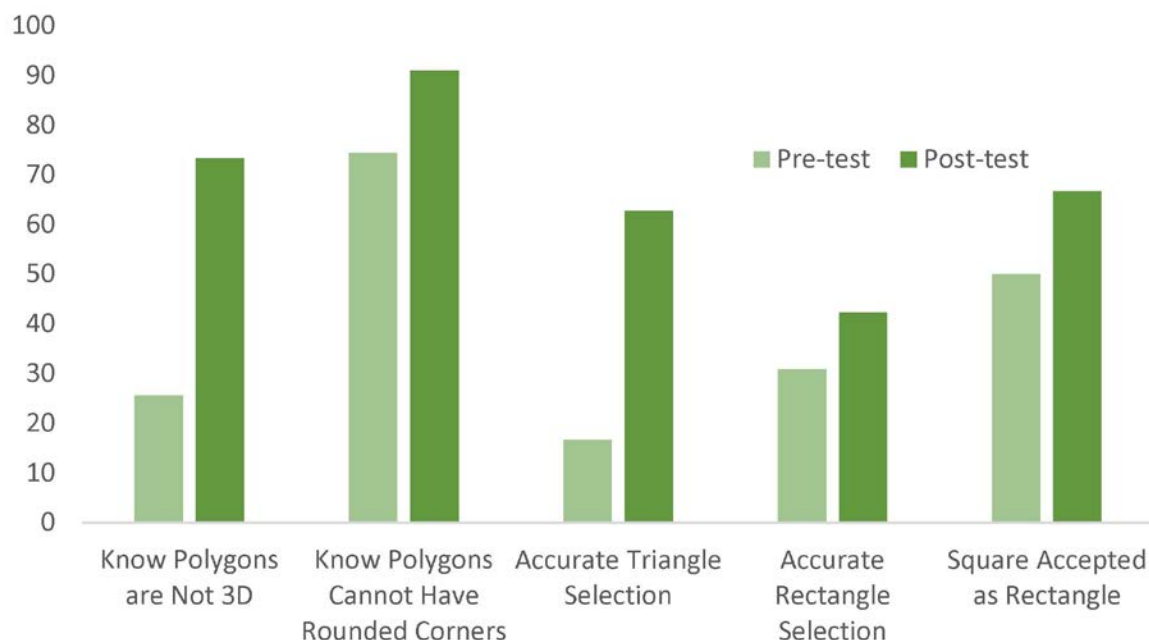


Figure 2
Content Question Accuracy by Objective

Polygons are 2D and have Straight Sides. At pretest only 25.6% of participants correctly discerned 3D distractors as not being polygons. After intervention 73.3% PSTs correctly discriminated between 2D and 3D options. A McNemar's test found this change was significant ($p < .001$). Similarly, at posttest more PSTs (91%) accurately recognized that shapes with rounded corners are not polygons compared to pretest (74.4%). A McNemar's test found this improvement was significant ($p = .002$).

Polygon Accuracy. To assess PST pre-post conceptions of triangles and rectangles, McNemar's tests were conducted. PSTs were significantly ($p < .001$) more likely to correctly categorize non-triangles from triangles at posttest (62.8%) than at pretest (16.7%). PSTs were more likely to recognize rectangles accurately overall on posttest (42.3%) than pretest (30.8%); however, this difference was not statistically significant ($p = .093$). After intervention, PSTs were significantly more likely ($p = 0.011$) to recognize the rotated square as well as horizontally oriented square as valid rectangles from pre (50%) to posttest (66.7%). Looking only at the valid quadrilateral items, we analyzed whether PSTs selected quadrilaterals with fewer than four right angles as rectangles. Surprisingly, more PSTs accepted quadrilaterals without right angles at posttest (34.6%) than pretest (24.4%), although the McNemar's test showed that this difference was not significant.

Discussion

It is rare, particularly in a course where students have high anxiety about the subject, to ensure activities are engaging, mathematically productive, and perceived as such by the majority of PSTs. Yet, this instructional activity that was atypical for this content course seems to be a promising approach to disrupt typical PST conceptions of geometric shapes. In PSTs' own words critiquing the accuracy of shapes in children's books was *interesting, fun, critical, helpful,* and

eye-opening (revisit Figure 1). The degree of PST buy-in (98% felt *worthwhile*) together with their perception that it improved their CCK (90% said improved knowledge; 83% said revealed a misconception) suggests strong evidence for the value of using this activity in a content course.

The content assessment items demonstrated that this activity was most successful in changing PSTs' conceptions of shapes by correcting their conflation of 2D with 3D shapes. This short activity quickly and significantly ($p < .001$) improved this crucial aspect of PSTs' shape concepts such that 73.3% of PSTs correctly disregarded 3D distractors. This was twice the proportion of PSTs who were accurate in the only study we found that assessed 2D-3D conflation of shapes (Luneta, 2014). Moreover, the activity fostered changes in visual to descriptive level thinking with three times as many PSTs now agreeing that shapes have to be identified based on properties, not visual appearances. Furthermore, unlike any other studies, our assessment included rounded corner distractors for two polygons to assess whether PSTs know that polygons must have straight sides that meet at vertices. Almost all of the PSTs (91%) demonstrated this critical understanding after the activity. Finally, an important purpose of this activity was to help PSTs shift from a partitive definition of a rectangle to a hierarchical definition (De Villiers, 1994). This 40-minute activity fostered significant improvement towards this goal ($p = .011$). Although it was disappointing that one-third of the PSTs would need additional instruction to understand a square is a type of rectangle, these results were five times better than the accuracy Fujita and Jones (2007) reported. It was surprising then that improvement on the rectangle accuracy task was not statistically significant. Further item analysis, however, revealed that some PSTs who broadened their conception of rectangle to include quadrilaterals with all equal sides, also released the critical property of right angles. Thus, this informs instructors that it would be crucial to emphasize the meaning of the term *rectangle* as *right-angle* by ensuring all PSTs attend to this explanation in the rating scale (see Nurnberger-Haag, 2018, Appendix B).

Conclusion

This study informs researchers as well as instructors of courses that focus on the undergraduate mathematics preparation of teachers in several ways. First, although it is commonly known that elementary and special education PSTs have less content knowledge than appropriate and that many themselves feel this way about mathematics overall, shapes are often still perceived as simple content they already know. Consequently, this study reveals that instructors and researchers should pre-assess PST shape knowledge to inform content course instruction. Second, these PSTs had passed sufficient courses and math assessments to gain entry to the content course, so by this time PSTs should know all the information assessed here. Yet, prior to the activity they did not. Typical instructional approaches in content courses have meant that PSTs continue to carry their incomplete shape conceptions into methods courses and their teaching careers (van der Sandt & Nieuwoudt, 2003). Thus, proceeding to try to correct such engrained misconceptions with instructional practices similar to prior instruction (i.e., completing mathematics problems) would be unlikely to lead to different results. In contrast, as a way to encourage PSTs to realize their own misconceptions, PSTs used a rating scale that a) identified common errors found in children's books, b) explained why these are errors and c) provided accurate property information. Thus, although atypical for a mathematics content course, this 40-minute activity that PSTs almost universally valued and significantly improved shape content knowledge seems to be an innovative way to disrupt PSTs' typical conceptions. Moreover, PSTs simultaneously described the activity in positive ways such as "Fun!" Consequently, these results have strong practical significance, particularly in courses in which

PSTs usually find it uncomfortable to admit they do not know something they should have mastered years prior.

To find ways to facilitate PSTs to overcome incomplete and inaccurate geometric concepts entrenched since they encountered their first board book, game or toy (Nurnberger-Haag, 2017; Resnick, Verdine, Golinkoff, & Hirsch-Pasek, 2016) will likely take such innovative instruction. A detailed explanation of how to implement the activity is in Nurnberger-Haag et al. (2020) using the open source rating scale (Nurnberger-Haag, 2018). Future research should explore in what way and for what reasons finding fault with published materials such as children's books frees PSTs to acknowledge and possibly feel better about their content knowledge while adopting a disposition to grow through such mathematics work. Interviews or focus groups could be used to uncover to what extent PSTs and instructors perceive such an approach as atypical and relative benefits when the topics learned are those that have the most entrenched inaccuracies. This study demonstrated that activities that encourage PSTs to apply mathematical ideas to critique publicly available resources should be recognized in mathematics departments as valid mathematical learning experiences. The results shared here suggest that approaches that are atypical for mathematics content courses may be productive launching points to initiate shifts in PST thinking and provide unique shared experiences on which subsequent more typical instruction can continue to build. Research is needed to develop and test use of atypical approaches to alter the most robust and long-standing conceptions of those who will teach the next generation.

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