IUMPST: The Journal. Vol 1 (Content Knowledge), February 2021 [www.k-12prep.math.ttu.edu]
ISSN 2165-7874

REAL ANALYSIS MATHEMATICAL KNOWLEDGE FOR TEACHING: AN INVESTIGATION

Blain Patterson
Virginia Military Institute
pattersonba@vmi.edu

The goal of this research was to investigate the relationship between real analysis content and high school mathematics teaching so that we can ultimately better prepare our teachers to teach high school mathematics. Specifically, I investigated the following research questions. (1) What connections between real analysis and high school mathematics content do teachers make when solving tasks? (2) What real analysis content is potentially used by mathematics teachers during the instructional process?

Keywords: Teacher Content Knowledge, Real Analysis, Mathematical Understanding for Secondary Teachers (MUST)

Introduction

What do mathematics teachers need to know to be successful in the classroom? This question has been at the forefront of mathematics education research for several years. Clearly, high school mathematics teachers should have a deep understanding of the material they teach, such as algebra, geometry, functions, probability, and statistics (Association of Mathematics Teacher Educators, 2017). However, only having knowledge of the content being taught may lead to various pedagogical difficulties, such as primarily focusing on procedural fluency rather than conceptual understanding (Ma, 1999). The general perception by mathematicians and mathematics educators alike is that teachers should have some knowledge of mathematics beyond what they teach (Association of Mathematics Teacher Educators, 2017; Conference Board of the Mathematical Sciences, 2012, Wasserman & Stockton, 2013). The rationale being that concepts from advanced mathematics courses, such as abstract algebra and real analysis, are connected to high school mathematics content (Wasserman, Fukawa-Connelly, Villanueva, Meja-Ramos, & Weber, 2017). For example, the field axioms discussed in abstract algebra surface as the properties of equality in high school mathematics when solving equations (Wasserman, 2016).

If explicit connections exist between the content of real analysis and high school mathematics, it is essential that those who teach calculus, precalculus, and algebra have a firm understanding of this subject (Wasserman et al., 2017). However, there exists little research on the connection between learning real analysis and teaching high school mathematics, despite findings from various studies that assert student achievement is related to the content knowledge of their teachers (Ball, Thames, & Phelps, 2008; Begle, 1972; Hill, Rowan, & Ball, 2005; Monk, 1994). These studies focus on either pedagogical content knowledge or the relationship between knowing abstract algebra and teaching high school algebra. Although these studies have made significant contributions to our understanding of the need for strong content knowledge for teachers, we still know very little about the relationship between learning real analysis and teaching high school mathematics. Therefore, the goal of this study was to investigate this relationship so that we can ultimately better prepare teachers to teach high school mathematics.

Literature Review

The Mathematical Understanding for Secondary Teachers (MUST) framework developed by Heid, Wilson, and Blume (2015) consists of the combination of the mathematical proficiency, mathematical activity, and mathematical context perspectives of teaching mathematics. Mathematical proficiency includes conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, as well as historical and cultural knowledge. Engaging in mathematical activity can be thought of as doing mathematics, with the emphasis is on those mathematical activities that teachers employ and that they want their students to learn (Heid, Wilson, & Blume, 2015). This includes activities such as mathematical noticing, reasoning, and creating. Unlike mathematical proficiency and activity, which are present in a variety of scientific fields, mathematical context can be described as understanding how students think about mathematics (Heid, Wilson, & Blume, 2015). In addition to being able to know and do mathematics, teachers must also be able to facilitate the development of their student's mathematical proficiency and activity. For example, this may include probing mathematical ideas, understanding the mathematical thinking of students, knowing and using the curriculum, and assessing the mathematical knowledge of students (Heid & Wilson, 2016). According to Heid and Wilson (2016), the combination of the mathematical proficiency, mathematical activity, and mathematical context perspectives together form a picture of the mathematical knowledge required to teach high school. Moreover, understanding according to the MUST framework is not simply the sum of knowing mathematics and knowing how to teach. The task of teaching mathematics cannot be partitioned into such simple categories; rather, teaching mathematics requires a unique combination of the two.

According to the *Standards for Preparing Teachers of Mathematics* developed by the Association of Mathematics Teacher Educators (2017), all teachers should possess robust knowledge of both mathematical and statistical concepts that form a foundation of what they teach. Additionally, all teachers should have pedagogical knowledge, including effective and equitable teaching practices as well as a firm understanding of how students think about mathematics (Association of Mathematics Teacher Educators, 2017; Conference Board of the Mathematical Sciences, 2012). For high school mathematics teachers, this means they must have a deep understanding of single- and multivariable calculus, probability and statistics, abstract algebra, real analysis, modeling, differential equations, number theory, and the history of mathematics. Although each of these subjects may be of equal importance, the focus of this study will be to investigate teacher knowledge of real analysis in relation to classroom teaching.

Real analysis is a course that nearly all mathematics majors and some mathematics education majors are required to take (Conference Board of the Mathematical Sciences, 2012). Standard topics covered in real analysis include the real number system, functions and limits, topology of the real numbers, continuity, differential and integral calculus for functions of one variable, infinite series, and uniform convergence (Bartle & Sherbert, 2011). This course is often viewed by pre-service high school mathematics teachers as daunting and disconnected from practice (Goulding, Hatch, & Rodd, 2003; Wasserman, Villanueva, Mejia-Ramos, & Weber, 2015). However, perceptions of this disconnect are incorrect since there are many explicit connections between what is learned in an introductory real analysis course and what is taught in high school mathematics courses (Wasserman et al., 2017).

Students in real analysis study the structure of the real number line and its subsets (Bartle & Sherbert, 2011). Analogously, high school mathematics teachers must develop student conceptions of the real number system as early as algebra. The convergence of sequences, which

ISSN 2165-7874

is studied rigorously in real analysis, also appears in the high school curriculum. Those who teach precalculus need to have a firm understanding of limits and continuous functions, both of which are studied intensely in an introductory real analysis course. Concepts that play a major role in calculus, such as differentiation, integration, and infinite series, make up the standard curriculum for the typical real analysis course (Bartle & Sherbert, 2011). Since these topics are foundational in the study of calculus, one may argue that calculus teachers ought to have a background in real analysis. However, let us consider the average high school mathematics teacher, who teaches courses such as algebra, geometry, and precalculus (not calculus). How can taking a course in real analysis benefit this teacher?

Methods

The purpose of this study was to investigate the relationship between knowledge of real analysis and classroom teaching in order to better understand how studying advanced mathematics can help improve and support the development of high-quality secondary mathematics teachers. By doing this, we may be able to determine if it is worthwhile for teachers to take advanced mathematics courses, and if so, what specifically about these courses is useful for teachers. This study serves as an initial step of this goal, by investigating the connections between real analysis and high school mathematics and how these connections can inform classroom teaching. Specifically, the following research questions will be addressed.

1. What connections between real analysis and high school mathematics content do teachers make when solving tasks? How can these connections be characterized in terms of *mathematical proficiency* and *mathematical activity*?

2. What real analysis content is potentially used by mathematics teachers during the instructional process? How can the use of this content knowledge be characterized in terms of *mathematical proficiency*, *mathematical activity*, and *mathematical context*?

This study took place over the course of the 2018-2019 academic year in the United States. Four teachers were asked to participate in this study; this is the recommended maximum number of cases to consider according to Creswell (2013), since considering more cases only reduces the depth of the analysis. A convenience sample of teachers was chosen since the first four teachers who meet these criteria were chosen to participate in the study Each of the four participants were in-service high school mathematics teachers who had previously taken a college-level course in real analysis. The teachers in this study taught at various high schools and have taken real analysis at various universities during either their undergraduate or graduate studies. Although their academic experiences differed, it was assumed that each teacher's real analysis course consisted of the following major topics: topology of the real numbers, sequences and series, limits, functions, differentiation, and integration.

To address the first research question, two task-based interviews were conducted with each of the four in-service teachers who are the cases in this study are and had taken a course in real analysis. Each interview lasted approximately one hour and video was recorded. One camera was used to record video, which was pointed directly at the paper the teacher was writing on. Interviews were conducted one on one in a quiet office to maximize the video and sound quality of the recording. In these interviews, the teachers solved problems that required them to either build up from or step down to practice (Wasserman et al., 2017). Teachers were provided with paper and writing utensils to illustrate their solution to each problem. These problems focused on the following domains: topology of the real numbers, sequences and series, limits, and functions.

For each of the tasks, participants were asked to solve a high school mathematics problem, shown a solution which involves concepts from real analysis, asked to reflect and explain the solution back to the researcher, asked how the connection made in this problem could inform their teaching, and asked how their real analysis course could have helped them integrate this connection into their teaching. Prior to the start of each interview, teachers were provided with the following description of the topics covered in real analysis to remind them of what they may have learned in the course: real number system, functions, limits, topology on the real line, continuity, differential and integral calculus for functions of one variable, infinite series, and uniform convergence.

In order to address the second research question, teachers were observed in their classrooms. The goal of these observations was to study how the real analysis concepts they have discussed as influencing their thinking might inform their teaching. Each of the four teachers was observed twice, where they chose lessons for me to observe that they believed required knowledge of real analysis. This was done so that potential connections between real analysis and high school mathematics would be observed. Prior to each observation, teachers submitted a detailed lesson plan and any accompanying materials used during the lesson. Observations lasted the entire class period, approximately 90 minutes for each observation. During these observations, video was recorded using only one camera and detailed field notes were taken. In addition to detailed accounts of what occurred during the lesson, information such as the course, number of students, and desk arrangement was recorded. One camera was used to record video from the back of the classroom so that the teacher could be observed and students' faces were not visible.

During the observations, the researcher had a passive role as an observer, focusing only on recording detailed field notes of the lesson. To supplement what is captured by the video recordings, it was essential that the field notes taken be immensely detailed. Recording field notes helped to provide detailed descriptions of events that cannot be captured using video recording, such as interactions between the teacher and students and between students. These thick, rich descriptions provided the foundation for the analysis of qualitative data (Patton, 2014). Field notes were recorded chronologically, that is, descriptions of events were recorded as they occurred in real-time. Notes were recorded every time the teacher presented a new concept, posed a question, had an interaction with students, etc. In addition to providing detailed descriptions of events that have occurred, field notes can be used to improve the trustworthiness of a study (Cowrin and Clemens, 2012). To enhance the trustworthiness in this study, excerpts from the transcripts and field notes as well as emergent findings were shared with the participants. The teachers who were observed were asked if these data sources and initial findings reflected their experiences accurately.

During the first phase of coding of the transcripts of the observations, a categorization was done on blocks of text from the transcripts, field notes, and lesson plans to categorize the teacher's practice as *mathematical proficiency*, *mathematical activity*, or *mathematical context*, as described by the MUST framework (Heid, Wilson, & Blume, 2015). Again, a block of text could consist of a word, phrase, sentence, or even several sentences. If a teacher demonstrated procedural fluency, conceptual understanding, strategic competence, adaptive reasoning, productive disposition, or any combination of these, then the classification *mathematical proficiency* was used. If a teacher demonstrated mathematical noticing, mathematical reasoning, mathematical creating, or any combination of these, then the classification *mathematical activity* was used. Finally, if a teacher probed mathematical ideas, understood the mathematical thinking of students, knew or used the curriculum, assessed the mathematical knowledge of learners, or any combination of these, then the classification *mathematical context* was used. This categorization was not used in the task-based interviews, since this could not be observed while solving purely mathematical tasks. Once all transcripts, lesson plans, and field notes were classified according to this scheme, the second phase of coding involved identifying emergent codes for *mathematical proficiency*, *mathematical activity*, and *mathematical context*. These emergent codes were then clustered and reduced to three themes for *mathematical proficiency*, two themes for *mathematical activity*, and three themes for *mathematical context*.

Results

Throughout the task-based interviews, teachers were able to make connections between metric spaces and coordinate geometry, convergence and equivalent representations of numbers, sequences and functions, and mappings and functions. To address the second part of this research question, data collected from the task-based interviews was analyzed using the MUST framework (Heid, Wilson, & Blume, 2015). Throughout the interviews, teachers engaged in similar proportions of *mathematical proficiency and mathematical activity*, with 35 total occurrences of each. In terms of *mathematical proficiency*, the following broad themes emerged: definition and algorithm. Teachers used their knowledge of real analysis to adapt and apply knowledge of a rigorous definition related to metrics, convergence, or functions to solve a problem and know why an algorithm or procedure related to limits, sequences, or functions worked. These themes align with the conceptual understanding, procedural fluency, and adaptive reasoning aspects of *mathematical proficiency* as described by Heid, Wilson, and Blume (2015). In a real analysis course, teachers would have developed a rigorous understanding of various mathematical definitions and concepts, which can then be applied to make sense of high school mathematics concepts. For example, a teacher may learn the definition of a function as a relation that maps each element in the domain to exactly one element in the range. They may even learn that a function is injective if every element of the range is the image of at most one element from the domain. Consequently, this may help teachers develop a deeper understanding of the graphical representation of a function. Moreover, this conceptual understanding of functions as mappings and injectiveness can then help them to better understand concepts and procedures from high school mathematics, such as applying the vertical line test to determine if a relation is a function or applying the horizontal line test to determine if a function is invertible.

In terms of *mathematical activity*, the following broad themes emerged: inductive reasoning, justifying, and modifying. Teachers used their knowledge of real analysis to help them to use inductive reasoning to make a conjecture or generalization about convergence, justify a claim about convergence or functions numerically, graphically, or using a formal proof, and use alternative mathematical representations of a sequence or function to reduce the complexity of the problem. These themes align with the reasoning when conjecturing and generalizing, justifying/proving, representing, and modifying/transforming/manipulating aspects of *mathematical activity* as described by Heid, Wilson, and Blume (2015). While studying real analysis, teachers are afforded the opportunity to use inductive reasoning to make conjectures (conjecturing and generalizing) and then prove these conjectures (justifying/proving) using definitions and theorems learned throughout the course. Additionally, teachers may use various mathematical representations to help them make sense of and solve problems (representing and modifying/transforming/manipulating). For example, one may make the conjecture that the rational numbers are countable, based on the fact that the natural numbers are countable. This claim can then be proven using known properties of the natural numbers and set theory. Although a formal proof using theorems and definitions would suffice, one may make sense of the proof by representing a diagonal mapping between the rational numbers and natural numbers.

Throughout the observations, teachers potentially made connections between how sequences, series, functions, and limits are studied in real analysis versus how these concepts are taught in high school mathematics classrooms. The second part of this research question was addressed by analyzing data collected from the teacher observations using the MUST framework (Heid, Wilson, & Blume, 2015). Throughout the two observations of each of the four participants, knowledge of real analysis impacted teachers' mathematical proficiency and mathematical context more than their mathematical activity, with 58 occurrences of *mathematical proficiency*, 60 occurrences of *mathematical context*, and 29 occurrences of *mathematical activity* in total. In terms of *mathematical proficiency*, the following broad themes emerged: language, connections, and application. Teachers used their knowledge of real analysis to use precise mathematical language or introduced and unpacked mathematical notation related to sequences, series, functions, or limits, explain or summarize mathematical connections between sequences and functions or limits and asymptotes, and use knowledge of sequences series, functions, or limits to explain how and why to apply a procedure, definition, or mathematical property to solve a problem. These themes again align with the conceptual understanding, procedural fluency, and adaptive reasoning aspects of *mathematical proficiency* as described by Heid, Wilson, and Blume (2015). The use of precise mathematical language and notation is essential in an advanced mathematics course such as real analysis. For instance, when discussing the notion of arbitrarily close, the δ-ε definition of a limit is used to formalize the notion of the limit of a function. This can be done by linking the $\boldsymbol{\delta}$-ε definition to a graphical representation. Additionally, studying real analysis may help teachers to improve their ability to work with and apply definitions, theorems, and other mathematical concepts to solve new problems. For example, the algebraic properties of limits of sequences can be proven by simply using the fact that sequence can be defined as functions along with the algebraic properties of limits already derived for functions.

In terms of *mathematical activity*, the following broad themes emerged: representing and verifying. Teachers used their knowledge of real analysis to use an example of an equation, graph, or other representation to illustrate a concept related to sequences, limits, or functions and justify a claim or derive a formula or property related to sequences, series, functions, or limits using definitions, properties, or examples. These themes align with the representing and justifying/proving aspects of *mathematical activity* as described by Heid, Wilson, and Blume (2015). Teachers engaged in a subset of the aspects of mathematical activity that they demonstrated during the task-based interviews, but in different ways. In the task-based interviews, teachers used multiple representations so that they could make sense of and solve a problem. In the classroom, however, these various mathematical representations were used to illustrate a concept to their students. For example, two of the teachers presented both equations and graphs of arithmetic and geometric sequences along with linear and exponential functions to show the similarities and differences between the two mathematical objects. In terms of the justifying or proving aspect of mathematical activity, it was common for teachers to derive known formulas using various algebraic properties. This differs from how teachers verified claims in the task-based interviews in that teachers would make conjectures and attempt to verify their claims, which turned out to be an iterative process in nature. As mentioned in the discussion

ISSN 2165-7874

of *mathematical activity* demonstrated during the task-based interviews, studying real analysis can afford teachers the opportunity to engage in this type of mathematical thinking.

Finally, in terms of *mathematical context*, the following broad themes emerged: questioning, building, and curriculum. Teachers used their knowledge of real analysis to answer and ask questions related to sequences, series, functions, or limits using precise and accurate mathematical language and notation, assess and build on prior student knowledge of sequences, series, functions, or limits, and select problems and tasks that help students build an understanding of sequences, series, functions, or limits. These themes align with the probe mathematical ideas, access and understand the mathematical thinking of learners, know and use the curriculum, and assess the mathematical knowledge of learners aspects of *mathematical context* as described by Heid, Wilson, and Blume (2015). Because of their extensive knowledge of the real numbers, sequences, and functions developed in their real analysis courses, teachers were able to ask their students meaningful questions (probe mathematical ideas) as well as answer student questions using precise mathematical language and notation (access and understand the mathematical thinking of learners). For example, this was observed when one teacher explained to her students that arithmetic sequences are a subset of linear functions, where the domain is restricted to the natural numbers. This was prompted by a student question about the relationship between the slope of linear functions and the common difference of arithmetic sequences. Additionally, teachers were able to use their knowledge of real analysis to select appropriate problems or tasks that could afford their students to make meaningful connections (know and use the curriculum and assess the mathematical knowledge of learners). Teachers' extensive knowledge of functions, sequences, and limits from real analysis helped them to develop and implement engaging tasks that promoted rich mathematical discussions. This is also consistent with previous findings that claim that horizon content knowledge has a positive impact on teachers' planned classroom practices (Wasserman & Stockton, 2013).

Conclusion

Similar to the results of the work done by Wasserman et al. (2015), teachers in this study did believe that the connections between real analysis and high school mathematics that were prevalent in the tasks could inform their teaching but felt that these connections were not explicitly discussed in their real analysis course or that these connections could have been emphasized more. Additionally, participants also believed that teachers would benefit from a real analysis course designed specifically for teachers, which again aligns with previous work (Wasserman et al., 2015). However, there were a few slight contrasts between the results of this study and the work done by Wasserman and colleagues (2015). First, participants in this study were in general able to make significant progress on the tasks, arriving at valid solutions for nearly every task. Additionally, all participants in this study felt that the connections in each task had some relevance to their teaching. These differences could be attributed to the background of the participants or difficulty of the tasks in comparison to those from Wasserman et al.'s (2015) study.

This study adds to the work done on the ULTRA project by Wasserman et al. (2017) by applying the MUST framework developed by Heid, Wilson, and Blume (2015) to investigate the impact of the knowledge of real analysis on teaching practices, specifically in terms of their mathematical proficiency, activity, and context. The goal of the ULTRA project (Wasserman et al., 2017) was to design, implement, and assess and innovative real analysis course for pre-service and in-service mathematics teachers. The results from this dissertation study contribute to the body of work on knowledge of advanced mathematics by providing specific forms of knowledge of real analysis can take. This complements the work done by Wasserman et al. (2017) by investigating how studying real analysis using the current implicit model may impact teaching practices. This dissertation study could be extended by comparing the impact of real analysis on mathematical proficiency, activity, and context when teachers study real analysis using the current implicit model versus the bidirectional model proposed by Wasserman et al. (2017).

The results of this study may inform the design and instruction of advanced mathematics courses for teachers such as real analysis, abstract algebra, number theory, or geometry. The goal is that these courses for teachers are more connected to the practice of teaching high school mathematics. With respect to real analysis specifically, the results of the task-based interviews and teacher observations have pedagogical implications for high school mathematics teachers, mathematicians, and mathematics educators. However, in the likely event that a real analysis course is not offered specifically for teachers, but for mathematics majors in general, the connections between functions studied in real analysis and functions taught in high school mathematics should be explicitly discussed in pedagogy focused courses. Moreover, the model developed by Wasserman and colleagues (2017) could be implemented so that teachers are afforded the opportunity to build up to and step down from practice as it relates to the teaching and learning of functions. For example, Problem 4 of the task-based interviews required teachers to step down to the practice of teaching function concepts by starting with more abstract notions of what it means for a function to be invertible. For both pre-service and in-service high school mathematics teachers, this means that they could seek professional development opportunities that help them to better make connections between advanced mathematics and CCSS-M (2010) related to functions.

The results of both the task-based interviews and teacher observations also have meaningful pedagogical implications. During these interviews and observations, teachers demonstrated *mathematical proficiency* through their understanding of definitions and procedures, the use of precise mathematical language and notation, and the ability to make connections between mathematical concepts and apply these connections to solve problems. Teachers also engaged in *mathematical activity* by using inductive reasoning, justify claims, and representing mathematical objects in a variety of ways. These are all skills that all high school mathematics teachers should possess and promote for their students. Therefore, it is imperative that when studying real analysis, teachers be afforded ample opportunities to learn rigorous mathematical definitions, think inductively, justify claims, and use multiple representations.

For mathematicians and mathematics teacher educators, this means that they should work to design and structure courses focused on real analysis for teachers in a way that promotes the growth of these skills. This could be achieved through a variety of means, such as classroom instruction, problem sets, quizzes, exams, or projects. Again, if high school mathematics teachers do not complete a real analysis course specifically designed for them, it is crucial that teachers are given opportunities to study rigorous mathematical definitions, engage in inductive reasoning, justify claims, and use multiple representations in their pedagogy focused courses. Finally, teachers demonstrated *mathematical context* by questioning, assessing and building on prior student knowledge, and using the curriculum appropriately. For mathematics teacher educators especially, this means that teachers are provided the opportunity to engage in these aspects of *mathematical context* within their pedagogy focused classes and that the impact that knowledge of real analysis has on these aspects is explicitly addressed.

Going forward, I plan to use the results of this study to produce practical and concrete materials for mathematicians, mathematics teacher educators, and high school mathematics teachers. For mathematicians and mathematics teacher educators, I hope to develop curriculum materials for teachers, using research-based connections between real analysis and high school mathematics. This could take the form of an open-source textbook that could be easily shared with those teaching either real analysis courses for teachers or courses that focus on making connections between college and high school mathematics. Rather than a full textbook, this may also take the form of supplementary resources (e.g. modules) that mathematicians and mathematics teacher educators could use as needed in their classrooms. Although work like this is currently being during by Wasserman et al. (2017), they have not placed much emphasis on sequences and functions, which I have found to be essential components of the real analysis curriculum for teachers. For teachers, this may take the form of workshops to help them to access their knowledge of real analysis and apply it to concepts that they teach in their classrooms. This practical dissemination of the results from this study may be done at the local, state, or even national level.

Finally, I believe it would be worthwhile to study connections between other advanced mathematics topics (abstract algebra, number theory, geometry, etc.) and high school mathematics. This process of conducting task-based interviews, observing teachers in their classrooms, and then analyzing the data using the MUST framework (Heid, Wilson, & Blume, 2015) could be done with any advanced mathematics course that high school mathematics teachers are required to complete. Finally, rather than considering the MUST framework (Heid, Wilson, & Blume, 2015), one may also consider investigating the impact of knowledge of real analysis on axiomatic, logical, inferential, peripheral, and evolutionary knowledge as described by Stockton and Wasserman (2017).

References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Raleigh, NC.

Ball, D.,Thames, M. & Phelps. G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Bartle, R. G., & Sherbert, D. R. (2011). *Introduction to real analysis* (Vol. 2). New York: Wiley.

Begle, E. (1972). Teacher knowledge and student achievement in algebra. *School Mathematics Study Group Reports Number 9*. Palo Alto: Stanford University.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. Washington, D.C.: American Mathematical Society.

Creswell, J. W. (2013). *Qualitative Inquiry & Research Design: Choosing Among Five Approaches* (3rd ed.). Thousand Oaks, CA: Sage.

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics*, *61*(3), 293-319.

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of education for teaching*, *15*(1), 13-33.

Goulding, M., Hatch, G., & Rodd, M. (2003). Undergraduate mathematics experience: Its significance in secondary mathematics teacher preparation. *Journal of Mathematics Teacher Education*, *6*(4), 361-393.

Harel, G. (2008). A DNR perspective on mathematics curriculum and instruction. Part II: with reference to teacher’s knowledge base. *ZDM*, *40*(5), 893-907.

Heid, M. K., & Wilson, P. S. (2016). Framework for mathematical understanding for secondary teaching: A mathematical activity perspective. *Proceedings of the Nineteenth Annual Conference on Research in Undergraduate Mathematics Education (RUME XIX*). (pp. 218-227).

Heid, M. K., Wilson, P. S., & Blume, G. W. (Eds.). (2015). *Mathematical understanding for secondary teaching: A framework and classroom-based situations*. IAP.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American educational research journal*, *42*(2), 371-406.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Monk, D. H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review*, *13*(2), 125-145.

Patton, M. (2014). Thick description and case studies: The bedrock of qualitative analysis. *Qualitative Research & Evaluation Methods: Integrating Theory and Practice*. Thousand Oaks, CA: SAGE.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher*, *15*(2), 4-14.

Star, J. R., & Murray, E. (2013). What do secondary prospective mathematics teachers need to know? Content courses connecting secondary and tertiary mathematics. *Notices of the Ams*, *60*(10).

Stockton, J. C., & Wasserman, N. H. (2017). Forms of knowledge of advanced mathematics for teaching. *The Mathematics Enthusiast*, *14*(1-3), 575.

Wasserman, N. H. (2016). Abstract algebra for algebra teaching: Influencing school mathematics instruction. *Canadian Journal of Science, Mathematics and Technology Education*, *16*(1), 28-47.

Wasserman, N., & Stockton, J. (2013). Horizon content knowledge in the work of teaching: A focus on planning. *For the Learning of Mathematics*, *33*(3), 20-22.

Wasserman, N., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2015). Secondary mathematics teachers' perceptions of real analysis in relation to their teaching practice. In *Annual Conference on Research in Undergraduate Mathematics Education (RUME)*. (pp. 1052-1055).

Wasserman, N. H., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2017). Making Real Analysis Relevant to Secondary Teachers: Building Up from and Stepping Down to Practice. *PRIMUS*, *27*(6), 559-578.

Zazkis, R., & Leikin, R. (2009). Advanced mathematical Knowledge: How is it used in teaching? In *Electronic proceedings of the Sixth Conference of the European Society for Research in Mathematics Education (CERME-6)* (pp. 2366-2375).

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning*, *12*(4), 263-281.