DO PROSPECTIVE ELEMENTARY AND MIDDLE SCHOOL TEACHERS UNDERSTAND THE STRUCTURE OF ALGEBRAIC EXPRESSIONS?

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Abstract

A large number of students' mistakes in algebra are due to their inability to see the structure of a mathematical expression. This study analyzes and compares the typical mistakes made by prospective elementary and middle school teachers as these students progress through the courses at California State University at Long Beach. The study shows that the students have difficulties recognizing structures of algebraic expressions not only at the introductory level but also later as the students take calculus and senior level courses.

The need for a thorough understanding of the structure of an algebraic expression when performing mathematical operations has been recognized by a number of authors. Kirshner (1989) suggests that "the ability to comprehend the syntactic structure of an algebraic expression is fundamental to competent performance in algebra." In Yerushalmy (1992), one can find that "the ability to transform involves mastering of algebraic rules as well as analyzing structures of expressions."

Much research (Booth, 1989; Booth, 1999; Herscovics et al., 1995; Kieran, 1989; Kieran, 1999; Lodholz, 1999; Sfard, 1991; Sfard et al., 1994; Wagner et al., 1999b) shows that the difficulties in recognizing the structure of mathematical expressions are due to the different treatment of expressions in algebra and arithmetic. "In algebra, [students] are required to recognize and use the structure that they have been able to avoid in arithmetic" (Kieran, 1989). In arithmetic, mathematical expressions are treated from the operational point of view, as a command to perform operations, whereas in algebra, mathematical expressions are treated from the structural viewpoint, as an object of algebraic manipulation. "Abstract notions can be approached in two fundamentally different ways: structurally as objects, & operationally – as processes" (Sfard, 1991).

For instance, in arithmetic, the expression "$3+4$" is a "sum" of two numbers perceived by students as a command to perform addition of the two numbers (operational approach), while in algebra, the "sum" is the "name" of the expression (the structural approach). Thus, students should be aware of the importance of the treatment of mathematical expressions from both points of view (operational and structural), "… certain mathematical notions should be regarded as fully developed only if they can be conceived both operationally and structurally" (Sfard, 1991).

In mathematical textbooks, the operational meaning of the word "sum" is explained precisely. However, the structural meaning of it is explained only for expressions involving a single operation (for example, “$x+y$” is a sum). For expressions with more than one operation, there is no explanation. Students are not being taught that, for instance, “$2 \bullet 3+5$” is a sum as well.

According to Sfard (1991), the structural understanding of an abstract notion can be automatically acquired through much operational practice. However, at this time, students do not have enough operational practice with evaluating mathematical expressions because of the intensive use of calculators. Possibly, this is one of the reasons that at the present time students cannot “see” the structure of a mathematical expression, and have enormous difficulties with understanding symbolic language.

If students are to be adequately prepared for algebra – to transition successfully from arithmetic to algebra – they must have a foundation in elementary school that prepares them for conceiving a given arithmetic expression as a mathematical object as well as an operational process. It is anticipated that students who understand this concept in the elementary grades would approach algebra more confidently and perform more successfully.

In order to teach successfully algebraic language and algebraic thinking processes to beginners, teachers should have a full command of the subject themselves. If an elementary school teacher does not understand the structure of an expression in depth, his or her ability to communicate the concept to students is severely impaired.

The intent of the study is to test whether the structural representation of mathematical expression is understood by pre-service elementary and middle school teachers.

Methodology

In the study conducted at California State University, Long Beach (CSULB) in the Spring term, 2002, a diagnostic test was administered to 366 students. The test assessed the students' understanding of the terminology related to the structure of mathematical expressions and the syntax of algebraic language. The students were given one multiple-choice question and four free-response questions. In addition, the students were asked to explain their answers. The questions are as follows:

<u>Question 1.</u> What is the name of the expression $4x^2-9y^2$?
Choices: (a) difference of squares, (b) difference of products, (c) square of difference.

<u>Question 2.</u> If possible, cancel out the common factor in the expression $\frac{2+x}{2}$.

<u>Question 3.</u> If possible, simplify the expression $\frac{2x+2}{2x}$.

<u>Question 4.</u> Use the statement “If $x^2=25$, then $x=\pm 5$” to solve the equation $(x+1)^2=25$.

<u>Question 5.</u> Use the statement “ If $2x+3=5$, then $x=1$” to solve the equation $2(y+1)+3=5$.

The first question aimed at finding out whether the students understood the structure of the expression. The choices (a) or (b), difference of squares or difference of products, were considered the acceptable answers. In the second and third questions, students were supposed to use the fundamental property of fractions $\frac{ac}{bc}=\frac{a}{b}, b,c \neq 0,$ to cancel the common factor in the numerator and the denominator. Acceptable answers for the second question were "impossible" and "$1+\frac{x}{2}$." For the third question, "$\frac{x+1}{x}$," and "$1+\frac{1}{x}$" were regarded as correct. The last two questions dealt with students' ability to recognize a similarity in the structures of the equations and their ability to use the given statement wisely. The "ideal" solutions in these cases were "$x+1=\pm 5 \Rightarrow x=4$ or -6," and "$y+1=1 \Rightarrow y=0$," respectively.

The experiment was held in eight different classrooms. The responses were then combined into seven major groups according to the level of the courses the participants were taking. The purpose of administering the test to the different groups was to investigate whether the students' skills in performing simple algebraic manipulations improve as they take more mathematics courses towards their degrees. The choice of these particular courses was motivated primarily by the large sizes of the classes and the willingness of the instructors to conduct the testing. The total sizes and the description of the groups are summarized as follows:

Table 1. Description and size of the participating groups of students

Group	Name	Description	Size
1	Beginning Algebra	MATH 001 *Elementary Algebra and Geometry*	28
2	Intermediate Algebra	MATH 010 *Intermediate Algebra*	70
3	Introductory Real Numbers	MTED 110 (Math Education) *The Real Number System for Elementary and Middle School Teachers*	47
4	Finite Math	MATH 114 *Finite Math*	57
5	(Pre)calculus	MATH 117 *Precalculus* MATH 122 *Calculus I*	57
6	Calculus	MATH 123 *Calculus II*	35
7	Advanced Education Course	MTED 402 *Problem Solving Applications in Mathematics for Elementary and Middle School Teachers*	72

The Liberal Studies Program at CSU, Long Beach, offers an Integrated Teacher Education Program (ITEP) that prepares K-8 multiple subject teachers. To fulfill the concentration in mathematics requirement, students must take the following core courses: *Probability and Activities-Based Statistics* (MTED 105), *Real Numbers* (MTED 110), *Geometry and Measurements* (MTED 312), and *Problem Solving Applications* (MTED 402).

The Department of Mathematics and Statistics offers a B.S. in Mathematics degree with Option in Mathematics Education. This option is for students preparing to teach mathematics at the secondary school level. The math course sequence required for this degree includes *Calculus I,II and III* (MATH 122, 124, and 224), *Introduction to Linear Algebra* (MATH 247), *Number Theory* (MATH 341), *College Geometry* (MATH 355), *Ordinary Differential Equations I*

(MATH 364A), *Probability Theory* (MATH 380), *Statistics* (MATH 381), and *Introduction to Abstract Algebra* (MATH 444).

An appropriate score on the Entry-Level Math (ELM) requirement is a prerequisite for MATH 001 and MATH 010. About 20% of the students in these courses are future K-8 elementary teachers (Groups 1 and 2).
As mentioned in the above paragraph, MTED 110 and MTED 402 are the capstone courses for the K-8 pre-service teachers, who entirely populate these courses (Groups 3 and 7). MTED 110 serves as a prerequisite for MTED 402. Three years of high school mathematics is required for MTED 110.
MATH 114 is offered primarily to Business majors, so the percentage of math education students taking this course is not more than 5% (Group 4).
Calculus courses are mandatory for students preparing to teach mathematics at the secondary school level. However, the majority of students in MATH 117, 122, 123 are Engineering or Computer Science majors. Only about 20% of the course body is comprised of math ed students (Groups 5 and 6).

Results

Overall, the study reveals that students have a strong conceptual misunderstanding of the structures underlying the mathematical symbols. The table below gives the percentages of students who answered the questions correctly in each of the seven groups.

Table 2. Percentages of correct responses

	Question				
Group	1	2	3	4	5
1	78.5%*	25.0%	0%	0%	14.30%
2	92.9%	28.6%	10.0%	2.9%	5.7%
3	95.8%	44.7%	29.8%	10.7%	21.3%
4	91.2%	63.1%	42.1%	29.8%	33.3%
5	94.8%	75.4%	36.8%	26.3%*	26.3%
6	97.2%	82.9%	71.4%*	42.9%**	34.3%
7	87.5%	44.4%	22.2%	5.6%	18.1%

* ** Percentages differ significantly from the other percentages in the same column (according to Duncan's multiple range test with type I error a=0.1).

As shown in Table 2, the students performed uniformly poorly regardless of the question and the group they belonged to.

Next, we will consider each question separately, analyzing the answers the students gave and the types of mistakes they made.
Question 1 was a multiple-choice question, so the possible mistakes were picking (c), square of difference, or failing to answer. The summary of the percentages of mistakes made is given in

the table below. For comparison, the table also contains the percentages of correct answers (in boldface).

Table 3. Percentages of mistakes made in answering Question 1

	Answers			
Groups	**(a)**	**(b)**	(c)	left blank
1	**57.1%***	**21.4%**	17.9%*	3.5%
2	**80.0%**	**12.9%**	2.9%	4.2%
3	**76.6%**	**19.2%**	2.1%	2.1%
4	**77.2%**	**14.0%**	5.3%	3.5%
5	**79.0%**	**15.8%**	5.3%	0%
6	**82.9%**	**14.3%**	0%	2.8%
7	**50.0%***	**37.5%**	5.6%	6.9%

* See the footnote after Table 2

As seen from the table, the majority of students in each group chose the *difference of squares* as the name of the expression "$4x^2 - 9y^2$." One might think that the reason for this choice is that the students noticed that $4x^2$ and $9y^2$ are equal to $(2x)^2$ and $(3y)^2$, respectively. Therefore, as one might think, the students have skipped one step in their heads and come up with difference of *squares*. Indeed, some of them did. They wrote that they picked the difference of squares because "the square root of 4 is 2 and the square root of 9 is 3" (Group 4, Finite Math), or "both squares can be ($\sqrt{\ }$) as well as their numbers" (Group 4), or "it is a subtraction of two perfect squares" (Group 5, Precalculus). On the other hand, there were explanations of the following type "both variables, x and y, are squared" (Group 1, Beginning Algebra, and Group 4), or "they both have squares and a subtraction sign" (Group 2, Intermediate Algebra), or "it is $x^2 - y^2$, just has coefficients" (Group 5). The message is clear here: the students see only the squares of x and y, and ignore the fact that they are **multiplied** by coefficients. There were other types of responses that showed that students are unfamiliar with the notion of the structure of an expression. For instance, they circled the *difference of squares* because "it is two different variables" (Group 5), or "I have always heard that terminology" (Group 6, Calculus), or "it is the only phrase I have heard of " (Group 6), or "don't know why just sounds right" (Group 6). Moreover, not all the students chose the correct answer because they understood the structures of expressions well, but because "they have the same square and the products are different" (Group 2), or "they are two different variables" (Group 5). The person who selected the difference of products as the answer, explained "it can't be the difference of two squares because x and y are not the same number and therefore cannot be subtracted from each other while written in this form" (Group 7, Advanced Education Course).

Students' mistakes on Question 2 can be divided into four categories: (1) Cancelled the twos $\frac{\not{2}+x}{\not{2}}$ to get x or $1+x$ or $x/2$; (2) Made an equation $\frac{2+x}{2}=0$ and solved it getting $x=-2$; (3) Multiplied by $\frac{2}{2}$ but ended up multiplying by 2 only the numerator obtaining $4+2x$

(or, erroneously, $x+4$ or $2+x$ or $2(x+1)$); (4) Didn't attempt to do the problem or got unusual answers like $\frac{x+1}{x}$ or 1 or 3/2, etc. Notice that the mistakes indicate a poor knowledge of the structure of an algebraic expression and a poor command of the algebraic rules. Table 4 presents the percentages of responses in the four categories (along with the percentage of the correct responses).

Table 4. Percentages of mistakes made in answering Question 2

	Categories of Mistakes				
Group	(1)	(2)	(3)	(4)	**correct**
1	71.4%*	0%	0%	3.6%	**25.0%**
2	54.3%	1.4%	1.4%	14.3%*	**28.6%**
3	38.3%	6.4%	10.6%	0%	**44.7%**
4	26.3%	0%	5.3%	5.3%	**63.1%**
5	15.8%	0%	3.5%	5.3%	**75.4%**
6	8.6%	2.9%	2.8%	2.8%	**82.9%**
7	41.7%	4.2%	4.2%	5.5%	**44.4%**

* See the footnote after Table 2

From the table is it apparent that an overwhelming majority of students in all the groups have made a mistake in category (1) that shows their conceptual misunderstanding of the structural form of the expression. The students in Group 1, Beginning Algebra, had the most trouble with the question. This group had the lowest percentage who got the question right, and the largest percentage who made a mistake in category (1). However, a high percentage (82.9%) of the students in the Calculus group, Group 6, answered the problem correctly and only 8.6% made a mistake in category (1). This indicates that more practice in higher level mathematics brings understanding of algebraic language.

Some of the students who wrote the correct answer "impossible" also wrote correct explanations like "no common factors" (Group 5, Precalculus) or "2 is not a common factor of the numerator" (Group 5) or "the 2 does not factor in $(2+x)$" (Group 5), or "it is already simplified to the lowest terms" (Group 2, Intermediate Algebra). Some explanations, however, were wrong and made it plain that students have difficulties with symbolic language. For instance, some chose the correct answer "impossible" because "you don't know what x is" (Group 2, and Group 7, Advanced Education Course), or "the solution to the problem is not defined (ex., $\frac{2+x}{x}=?$)" (Group 7).

Question 3, even though, seemingly analogous to Question 2, turned out to be insurmountably difficult for some of the participants. Typical mistakes can be classified as

(1) Cancelled $2x$'s or 2's to obtain 2 or 3 or $\frac{2}{2x}$ or $\frac{2x}{2}$; (2) Said "impossible"; (3) Left blank; or (4) Wrote incomprehensible answers (totaling 28 varieties). The percentages of each type of mistakes are summed up below.

Table 5. Percentages of mistakes made in answering Question 3

	Categories of Mistakes				
Group	(1)	(2)	(3)	(4)	**correct**
1	60.7%*	25.0%*	0%	14.3%	**0%**
2	58.6%*	2.9%	7.1%	21.4%*	**10.0%**
3	31.9%	2.1%	4.3%	31.9%	**29.8%**
4	24.6%	8.8%	3.5%	21.0%	**42.1%**
5	21.1%	14.0%	8.8%	19.3%	**36.8%**
6	5.7%**	0%	2.9%	20.0%	**71.4%***
7	38.9%	11.1%	5.6%	22.2%	**22.2%**

* ** See the footnote after Table 2

The poorest performance was observed in Group 1, Beginning Algebra (none of the students answered the question correctly, and 60.7% used the cancellation rule improperly), while the students in the Calculus course (Group 6) did the best on this question. A significantly higher percentage (71.4%) of the Calculus students came up with the right answer and only 5.7% of them cancelled improperly.

As for Question 4, there were three kinds of typical mistakes: (1) Plugged $x = \pm 5$ into the equation $(x+1)^2 = 25$ to get contradictions 36=25 and 16=25; (2) Tried to solve the quadratic equation ignoring the given statement (sad to say, only two students managed to get the correct answer this way); (3) did not do the problem or wrote "impossible" or something else. The results are given below.

Table 6. Percentages of mistakes made in answering Question 4

	Categories of Mistakes			
Group	(1)	(2)	(3)	**correct**
1	32.1%	35.7%	32.2%	**0%**
2	34.3%	22.8%	40.0%	**2.9%**
3	31.9%	46.8%	10.6%	**10.7%**
4	21.1%	35.1%	14.0%	**29.8%**
5	17.5%	42.1%	14.1%	**26.3%**
6	11.4%	37.1%	8.6%	**42.9%**
7	26.4%	43.0%	25.0%	**5.6%**

Notice that none of the learners in Group 1, Beginning Algebra, did the problem correctly, while the Calculus students, Group 6, got the highest percentage of correct answers (42.9%). A large proportion of students in all the groups have committed the mistake of the first type, which shows their understanding of the need to use the symbolic pattern but a misunderstanding of the structure of the equation. Instructive was the reasoning behind the answers. For example, as an explanation for why it was impossible, in his opinion, to solve the problem at hand, one of the students in Group 7, Advanced Education Course, wrote "This is impossible to solve. If $x^2 = 25$, then $(x+1)^2 = 25$ is not solvable because no matter what x equals, positive or negative, then if you add one to it and square it, it would not equal 25." This stresses once again the students' lack of understanding when they are dealing with symbolic language.

Question 5 was easier for the participants but some of them were taken aback by the fact that two different variables, x and y, were used. They explained "Can't solve because no x in equation." (Group 3, Finite Math) or "I'm confused because the variable was changed from x to y." (Group 2, Intermediate Algebra). When working on this question, the students either
(1) Ignored the given statement and solved the equation in y directly; or (2) Left the space blank or wrote something incoherent. The results are below.

Table 7. Percentages of mistakes made in answering Question 5

	Categories of Mistakes		
Group	(1)	(2)	**correct**
1	32.1%*	53.6%*	14.3%
2	57.1%	37.2%**	5.7%
3	61.7%	17.0%	21.3%
4	52.6%	14.1%	33.3%
5	54.4%	19.3%	26.3%
6	57.1%	8.6%	34.3%
7	59.7%	22.2%	18.1%

* ** See the footnote after Table 2

A really low percentage of correct answers was observed in Groups 1 and 2 (Beginning and Intermediate Algebra) (14.3% and 5.7%). The most mathematically advanced Calculus Group (Group 6) did better (34.3%), even though the differences are not statistically significant. Notably, the students in Group 1 have made the smallest proportion of mistakes of the first type (trying to solve the equation in y), and the largest proportion of untypical errors.

Discussion

The conducted study has shown that college students perform unsatisfactorily in the manipulation of algebraic expressions. The questions on the test dealt with recognizing the structure of an algebraic expression (Questions 1, 4 and 5), and applying rules for cancellation of a common factor (Questions 2 and 3). The problem of conceptual misunderstanding of both topics is most severe at the novices' level and is still substantial in Calculus classes. From Table 2, the poorest performance is shown by Groups 1 and 2, which is not surprising since these students have failed the ELM. The future K-8 multiple subject teachers (Groups 3 and 7) did show slightly better results but not significantly better. Group 4 (predominantly Business majors) gave noticeably more correct responses than the previously-considered groups. Finally, for the pre-calculus and calculus students (Groups 5 and 6), the percentage of correct answers was the highest but still quite low. Consequently, if students are not taught algebraic structures and the language of algebra properly in elementary courses, the non-understanding will persevere throughout their studies and later in their careers.

Researchers recognize the need of a powerful method to teach the subject to elementary school teachers (Carpenter et al. 2000; CBMS, 2001; Kaput, 1995; Wagner et al, 1999a; Kieran, 1999). Unfortunately, researchers agree, no such method exists yet. As Kieran (1999) points out, "… it is not obvious how the use of symbol manipulators in the early stages of learning algebra can help students develop a structural conception of algebraic expressions. This is the question for future research." The current study once again underscores the need for this research.

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