

Developing Mathematical Content Knowledge for Teaching Elementary School Mathematics

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Abstract

In this paper the authors present three design principles they use to develop preservice teachers' mathematical content knowledge for teaching in their mathematics content and/or methods courses: (1) building on currently held conceptions, (2) modeling teaching for understanding, (3) focusing on connections between content knowledge and other types of knowledge. The authors share results of individual research projects and teaching approaches focusing on helping preservice elementary teachers develop such knowledge. Specific examples from different content areas (whole number, fractions, angle, and area) are discussed.

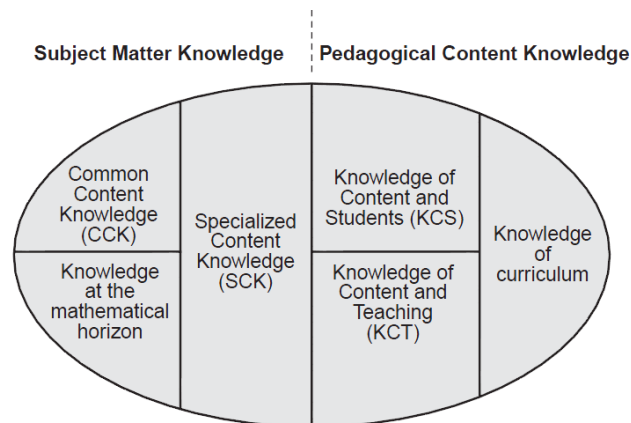
Key Words: Preservice Teacher Education; Teacher Education; Elementary; Mathematics; Content Knowledge; Mathematical Knowledge for Teaching; Pedagogy.

Introduction

The definition of the knowledge needed to teach mathematics has been the focus of recent discussions in the mathematics education community. Shulman (1986) was one of the initiators

of this discussion with the introduction of the idea of *pedagogical content knowledge*, which is the intersection of content specific knowledge and pedagogy. Ma (1999) used the phrase *profound understanding of fundamental mathematics* to identify the deep understanding of mathematics that teachers need. Hill, Ball, & Shilling (2008) introduced a framework for distinguishing between different types of knowledge included in the construct of *mathematical knowledge for teaching* (see Figure 1). This framework distinguishes between subject matter knowledge and pedagogical content knowledge. Subject matter knowledge is subdivided into common content knowledge, specialized content knowledge, and knowledge on the mathematical horizon. Pedagogical content knowledge is subdivided into knowledge of content and students, knowledge of content and teaching and knowledge of curricula. Hill, Rowan & Ball (2005) provide empirical support linking teachers’ mathematical knowledge for teaching to student achievement gains.

Figure 1:
Hill, Ball, & Shilling’s framework (2008, p. 377)



The authors of this manuscript find the mathematical knowledge for teaching framework to be useful when discussing different types of knowledge they want their preservice teachers to develop. However, the distinctions among the different types of knowledge appear blurry at times. Ball, Thames & Phelps (2008) acknowledge the problem of the fuzzy boundary among their six sub-domains. For example, Hill et al. (2008) described common content knowledge as “knowledge that is used in the work of teaching in ways *in common with* how it is used in any other professions or occupations that also use mathematics” (p. 377) and specialized content knowledge as “the mathematical knowledge that allows teachers to engage in particular *teaching* tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures and examine and understand unusual solution methods or problems” (p. 377). In recent years, several influential organizations such as the National Council of Teachers of Mathematics have called for a focus on conceptual understanding to help all students become successful mathematics learners (NCTM 2000). Thus, if a goal of instruction is that all mathematics learners are able to not only apply the correct procedures (procedural understanding) but also explain why the procedures work (conceptual understanding), the distinction between common content knowledge and specialized content knowledge becomes blurred. What would traditionally be considered to be only specialized content knowledge may become common content knowledge in the classroom that focuses on

learning with conceptual understanding. Similar issues of blurred boundaries arise when comparing other types of knowledge within this framework. For example, in the domain map of mathematical knowledge for teaching, knowledge of content and students resides in the pedagogical content knowledge area; however, in recent work (Philipp, et al. 2007) knowledge of content and students was used as a means to develop specialized content knowledge. Thus a connection to children's mathematical thinking may cross the boundaries between specialized content knowledge and pedagogical content knowledge.

A Framework for Design Principles for Preservice Elementary Teacher Mathematics and/or Methods Courses

While much progress has been made in developing a framework for mathematical knowledge for teaching, much work remains in the next step of determining what kinds of learning opportunities effectively help preservice teachers to develop such knowledge. Therefore our attention turns to the question of how we can use our current understanding of mathematical knowledge for teaching as a framework for designing content courses. To help preservice teachers develop mathematical knowledge for teaching, mathematics teacher educators first need to understand their students' currently held conceptions to be able to build on those. As the authors of *The Mathematical Education of Teachers* suggest, "the key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know" (CBMS 2001, p. 17).

The authors of this manuscript find that their attempts to build mathematical ideas from preservice teachers' currently held conceptions take two related approaches.

1. For some this means identifying what conceptions preservice teachers hold when they enter the classrooms and then building on those conceptions. Once the preservice teachers' initial conceptions are identified and it is understood how those conceptions develop, tasks can be created addressing those initial conceptions allowing for the development of more sophisticated ones.

2. Others support the development of preservice teachers' conceptions by limiting the mathematical ideas that can be used in explorations; only those ideas developed by the classroom community are allowed.

Both of these approaches can be employed simultaneously. For example, area formulae for polygons could be built on the general conceptions of area and shape properties the preservice teachers bring with them and then be developed as a class. The use of either single approach (or both in conjunction with each other), has been shown to promote construction of conceptual and procedural knowledge simultaneously while negotiating their mathematical understandings in the community of the classroom (cf. Cobb, Wood, and Yackel, 1990).

In teaching content courses for preservice teachers, the authors focus on engaging preservice teachers in developing their own understanding, facilitating opportunities for mathematical communication, and conducting formative assessments of their knowledge and development to inform instruction. In short, the authors model practices consistent with those described in the *Principles and Standards for School Mathematics* (2000). This type of experience is critical for preservice teachers not only so they learn the content of the course with understanding, but also so they can use it as a model for their own teaching.

Finally, the authors believe that content knowledge is connected to and supported by other types of knowledge. A goal of the mathematics teacher educators' practice is the development of connections between knowledge domains. Pedagogy employed by the authors in attempts to

achieve this goal includes explicating teacher moves (connecting to knowledge of content and teaching), using artifacts of children’s mathematical thinking (connecting to knowledge of content and children), and explaining curriculum decisions (connecting to knowledge of content and curriculum).

Thus in summary the authors share a framework of design principles that emerged from examining our work collectively:

1. Mathematical ideas are built on preservice teachers’ currently held conceptions.
2. Classes for preservice teachers should model teaching for understanding.
3. We focus on developing connections between content knowledge and:
 - Knowledge of content and teaching;
 - Knowledge of content and children;
 - Knowledge of curriculum.

In the next sections each author illustrates his or her personal application of this joint framework for designing preservice teacher courses at their respective institutions. The examples were chosen because they were connected to each author’s current investigative work and because they illustrate four different content areas essential in the elementary mathematics curriculum: place value, angles, the unit whole, and area.

Developing an Understanding of Multidigit Whole Numbers

Mathematical ideas are built on preservice teachers’ currently held conceptions. Many preservice teachers view the digits in a number in terms of ones rather than in terms of their value (Thanheiser, 2009b). They may see the 2 in 324 as 2 rather than as 20 or as 2 tens. Based on such a view of number, tasks were designed to help preservice teachers connect the digits to their representative value. One such task used digit cards (see Figure 2) and asked preservice teachers to use those cards to build numbers and operate on those numbers (add and subtract).

Figure 2.

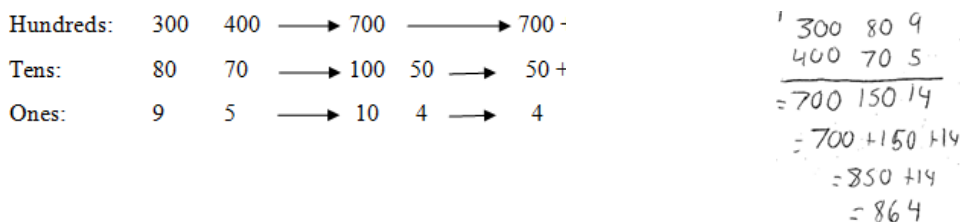
Digit cards representing the place value of each digit 0–9 (increments of 1, in green), 10–90 (increments of 10, in blue), and 100–900 (increments of 100, in red) and a representation of 423 with digit cards.



We teach our preservice teachers in the same way we want them to teach their classes. Students are asked to work on adding $389 + 475$ (among various addition tasks) using the digit cards. They do this on their own first, then discuss their approaches in small groups, and then share with the whole class. Most students are able to “invent” an expanded addition algorithm (see Figure 3).

Figure 3.

One preservice teacher's strategy to add $389 + 475$ (left) and another preservice teachers' written work (right)



In one study involving a class of 30 preservice teachers, all but one invented a form of the expanded algorithm (Thanheiser, 2009a). Work with children has shown that generating their own strategies deepens children's conception of number (Ambrose, 1998; Hiebert & Wearne, 1996); the same is true for preservice teachers (Thanheiser, 2009a).

These preservice teachers engage in creating their own knowledge of how their invented algorithms work and are therefore in a position to connect the symbols they write to the value of that symbol. One preservice teacher commented after using the cards:

[using the cards] shows that when you want to make a number, say 364, you need to get a 300 card, a 60 card, and a 4 card. It demonstrates that every number is composed of place values, and not just face values; it shows that 364 is not just a 3, 6, and 4 put together.

Develop connections between content knowledge and other kinds of knowledge. After students invented ways of adding and subtracting numbers using the digit cards they viewed video clips of children adding and subtracting numbers. One example was of a boy (2nd grade) using the expanded addition algorithm to add $274 + 368$. He first added $200 + 300 = 500$; $70 + 60 = 150$; and $4 + 8 = 12$ and then added the partial sums $500 + 150 + 12$ to get 642.

After viewing this clip one preservice teacher commented, "This way . . . is the easiest way for younger kids to add." Having connected the digits to their values (i.e. the 2 in 324 to 20) preservice teachers are now able to appreciate this algorithm. Another preservice teacher commented "It also gets them to focus on the concept of hundreds, tens, and ones, so they develop conceptual knowledge as opposed to simply procedural knowledge."

Connecting back to the definition of mathematical content knowledge. Using the example above, the common content knowledge could be identified as understanding and adding of three-digit numbers, the specialized content knowledge could be identified as using the cards to show why addition works, and the pedagogical content knowledge could be identified as discussing how children think about addition and how that impacts the preservice teachers' teaching.

Making Sense of Angle and Angle Measure

Mathematical ideas are built on preservice teachers' currently held conceptions. We have found preservice teachers have a difficult time articulating a clear definition of angle separate from an angle's measure or, sometimes, articulating any definition at all. To gain some perspective of their currently held conceptions of angles and angle measures, we initially ask students to respond to the following prompt: "Measuring an angle of a shape is different from

measuring a side of a shape because...” Some typical samples responses follow:

- “Measuring an angle of a shape is different from measuring a side of a shape because an angle is measured in degrees versus measuring a side is in length.”
- “Angles are measured using degrees within a radius. Sides are measured in lengths with increments.”

From these examples, we see that preservice teachers have some sense of sides having an *attribute* of length, but angles merely have degrees, a unit of measure. However, to develop a rich conceptual understanding of angle, we need for preservice teachers to be able to move beyond the unit of measure to really describe the *attribute* of angle that we are attempting to measure (i.e. the space between two rays) and to use that to begin to create a definition for angle.

We teach our preservice teachers in the same way we want them to teach their classes. We want our preservice teachers to develop a broader conception of angle, to be thought of not only in the traditional “two rays connecting at a common vertex” sense, but also focusing on measurable attributes such as the region of space between these two rays and, more dynamically, as a representation of a turn. Thus activities were designed to allow the preservice teachers to discover these different aspects of angle in the context of small group explorations as well as through whole class discussion of their emerging angle concepts, modeling a pedagogical approach for them to use with their own future students. We briefly describe the activities that attend to various components of these angle representations.

In the initial activity, students are given a piece of hamburger patty paper and are asked to “invent” a measuring device that would allow them to measure several angles (adapted from Wilson & Adams, 1992). Often this device takes the form of a paper “wedge” that mirrors characteristics of an angle. This “wedge activity” addresses several objectives: an experience with a non-standard unit of measurement (a “wedge”), recognition of the attribute being measured when finding an angle, and the development of the idea of the “degree” as a very small, “standard wedge” that fills the space in the angle. To develop the idea of angle as a representation of turn, we use two activities on the Texas Instruments TI-73 Explorer™ calculator; the “Angle” feature on the application SmileMath and a scaled-down version of Logo called “Logo Light”. Using SmileMath, students watch as an angle is created by one ray turning away from an initial, adjacent ray and either stop the turning motion to create an angle of an indicated measure or attempt to estimate the angle measure of a calculator-chosen angle. To then build on this turning notion, Logo Light provides an environment in which students can further their understanding of two-dimensional shapes and their properties while specifically focusing on angles as being created by “turtle turns”.

Develop connections between content knowledge and other kinds of knowledge. Sample sixth-grade student thinking (Browning & Garza-Kling, 2009) is shared with preservice teachers so they develop an idea of how children may respond to the same types of activities they themselves have participated in. For example, the sixth graders’ statements of “they had to of invented angles before protractors” and “their (sic) is more than degrees to an angle” help our preservice teachers realize the limitations in an angle definition that only mentions degrees. Preservice teachers are generally surprised by what the 6th graders say about angle since they

realize it is fairly close to their own thinking. Examining the children's angle conceptions allows preservice teachers to see the mathematical language children use to support their thinking, providing them with an opportunity to determine if that thinking is mathematically appropriate.

Connecting back to the definition of mathematical content knowledge. The angle activities above suggest common content knowledge would be identifying angles and describing their size using degrees. Specialized content knowledge would be knowing several ways to think of angle, depending upon the context in which it is used, understanding what degrees are and how to describe the unit of measure they provide, and knowing mathematically appropriate language and reasoning for children to use when developing their understanding of angle. The pedagogical content knowledge could be identified as understanding student misconceptions with angle and angle measure, such as thinking that the length of the rays affects the measure of the angle, and developing questions to guide the children's understanding of angle.

Developing Specialized Content Knowledge of the Unit Whole

Mathematical ideas are built on preservice teachers' currently held conceptions. Preservice teachers typically enter content courses with an incomplete understanding of rational numbers and the unit whole. Using the Learning Mathematics for Teaching instruments (Hill, Schilling, & Ball, 2004) a group of 244 preservice teachers were studied as they entered their methods course after having taken their content courses (Moss, 2006). A deeper look into the most frequently missed items, and examination of others' work (Ball, 1988 & Ma, 1999), provides evidence that understanding the unit whole with fractions is a big challenge. The most missed item was choosing correct representations of a unit whole. When choosing a representation for a quantity like $\frac{2}{3}$, choosing a correct unit whole seemed to be a challenge. While many of these preservice teachers understood it was appropriate to choose one square as your unit whole, or three squares as your unit whole, the participants did not see that choosing two squares as your unit whole and dividing them by three as an appropriate representation. Additionally, the third most missed item involved choosing a correct story problem to represent subtraction of fractions. For example, given the expression $\frac{1}{2} - \frac{1}{4}$ preservice teachers frequently incorrectly chose as the corresponding problem one such as "I have $\frac{1}{2}$ a jar of peanut butter and I eat $\frac{1}{4}$ of that. How much is left?" The preservice teachers often failed to realize that the unit whole was changing in the peanut butter example; a symbolic representation of that problem would be $\frac{1}{2} - \frac{1}{4}$ ($\frac{1}{2}$).

We teach our preservice teachers in the same way that we want them to teach their classes. Preservice teachers are asked to create various representations for $\frac{2}{5}$, $\frac{2}{6}$, and $2\frac{1}{4}$ using a range of manipulatives (such as Cuisenaire rods, base ten blocks, pattern blocks, fraction circles, fraction squares, measuring tape, link cubes, plastic animals, etc.). As they share their representations with the class, the variety of representations leads to a discussion of what the unit whole is in each, especially in relation to set models, linear models, and area models.

The concept of the unit whole is further developed by providing preservice teachers opportunities to deeply explore problems involving the unit whole (see Figure 4).

In the first problem, the unit whole is changing each time; in the second problem, every fraction refers to the same unit whole. Preservice teachers may initially attempt to solve the pearl problem exactly like the cookie jar problem, but soon realize the difference between the two as they begin using manipulatives to model the problem and have to change their representation.

Figure 4:

Two problems given to preservice teachers to explore the unit whole.

1. A cookie jar is sitting on the table. Megan comes along and eats $\frac{1}{2}$ of the cookies. Ella comes by later and eats $\frac{1}{3}$ of the remaining cookies. Tyler comes by and eats $\frac{1}{4}$ of the remaining cookies. Finally, Ross eats the remaining 6 cookies. How many cookies were in the jar to begin with?

2. A girl's necklace was broken during a waltz. One third of the pearls fell to the floor, one fifth rolled off the dance floor, one sixth were found by the girl, and one tenth were recovered by her dance partner. Six pearls remained on the string. How many pearls were on the necklace have it broke?

Develop connections between content knowledge and other types of knowledge. Children's thinking about fractions can enhance preservice teachers' understanding and provides a connection to knowledge of content and students. For example, Felisha, a third grader from the Integrating Math and Pedagogy videos (Philipp, 2005), is asked to solve the problem "If you have two cookies and want to share them with five friends, how much of a cookie would each friend get?" The preservice teachers solve the problem with a variety of manipulatives in front of them to help their reasoning and communication. Since some students believe the answer to be $\frac{2}{10}$ and others believe $\frac{2}{5}$, a discussion using a variety of models leads to the idea of $\frac{2}{10}$ of the plate of cookies, and $\frac{2}{5}$ of a cookie. After watching the video of Felisha, the preservice teachers notice and comment that she struggles with the same ideas.

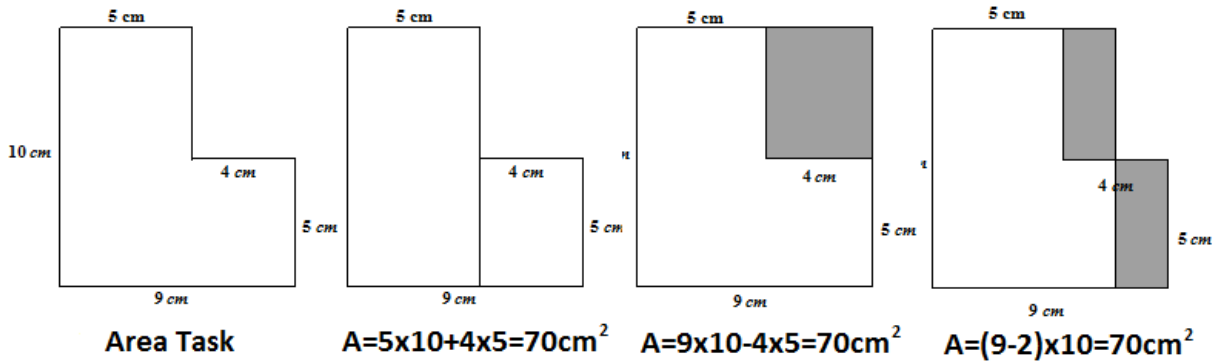
Connecting back to the definition of mathematical content knowledge. Using the examples above, the common content knowledge could be identified as understanding and solving problems involving fractions, the specialized content knowledge could be identified as understanding multiple representations of fractions and solving problems involving fractions with a focus on the unit whole, and the pedagogical content knowledge could be identified as understanding the struggles children have with the concept of the unit whole and how problems designed to help bring these issues to the surface can help the preservice teacher's understanding.

Developing Area Formulae Meaningfully

Mathematical ideas are built on preservice teachers' currently held conceptions. Virtually all preservice teachers have seen various area formulae. However, they were typically given those formulae and simply practiced their use on many figures. If a mathematics course is designed for the preservice teachers to actually experience deriving formulae in the way children might, they need to be restricted from using any idea that children might not know at that point. In the process of deriving the area formulae, the Area Task (Figure 5) - the exploration of finding the area of an L-shape - plays a central role.

Prior to this lesson, students have learned that the area of a rectangle can be calculated by multiplying its two dimensions as those measurements provide the number of unit squares that will fit along the dimensions; that is the only knowledge they can use at this point. Figure 5 shows three of the strategies preservice teachers typically develop during a lesson.

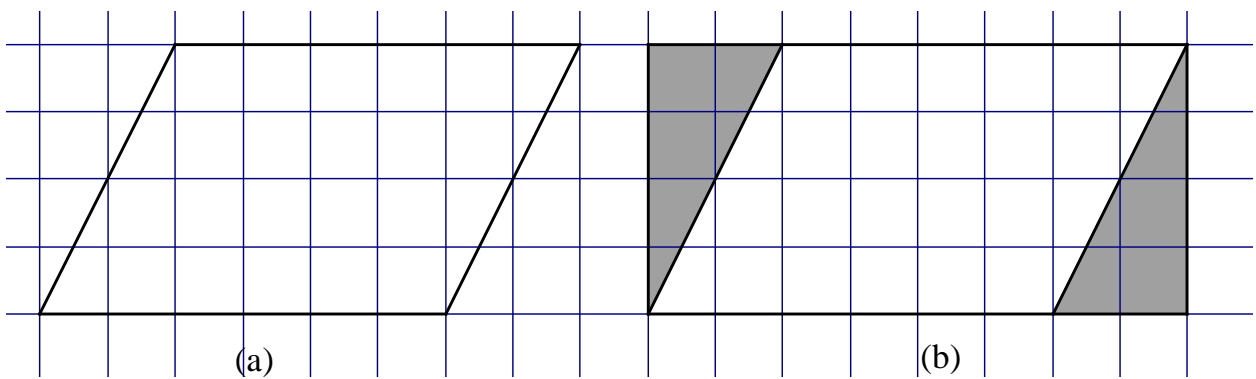
Figure 5:
Area Task and three typical solution strategies to the Area Task.



As the class, preservice teachers typically come up with 8 or 9 different strategies. After comparing and contrasting those different strategies, preservice teachers conclude that all strategies make use of familiar shapes, that is rectangles, at this stage. As a class, preservice teachers come to the understanding that if they are given an unfamiliar shape, that is, shapes for which they have yet to learn an area formula, they may still be able to calculate its area by making a familiar shape, or a collection of familiar shapes. Upon further examination of the strategies, preservice teachers group the strategies into three categories. Those categories are (1) sub-divide the given shape into a collection of familiar shapes, (2) make-it-bigger, and (3) cut and re-arrange the parts to form a familiar shape. The three methods shown in Figure 5 reflect these strategies.

With this knowledge, the class moves on to exploring other types of polygons. We next examine the area of parallelograms by initially exploring a parallelogram similar to the shape shown in Figure 6a. This is a very common activity in many of today's elementary school mathematics textbooks, however, after the previous discussion of 3 strategies, preservice teachers come up with a variety of strategies that are not often suggested in those textbooks (Figure 6b).

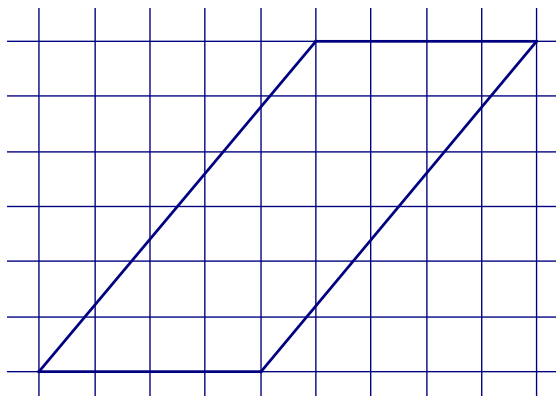
Figure 6. Parallelogram Area Task



Students are asked to find the area of this parallelogram (see Figure 6a) using only what they

have learned so far in class. Some students apply the Make-It-Bigger approach to find the area (see Figure 6b): Area of Parallelogram = Area of Large Rectangle - Area of Rectangle Made up of 2 Triangles. Next, the class is given another parallelogram as shown in Figure 7. Preservice teachers realize that some of the strategies used for the previous parallelogram do not work well with this parallelogram because it is "slanted too much."

Figure 7 "Slanted" parallelogram



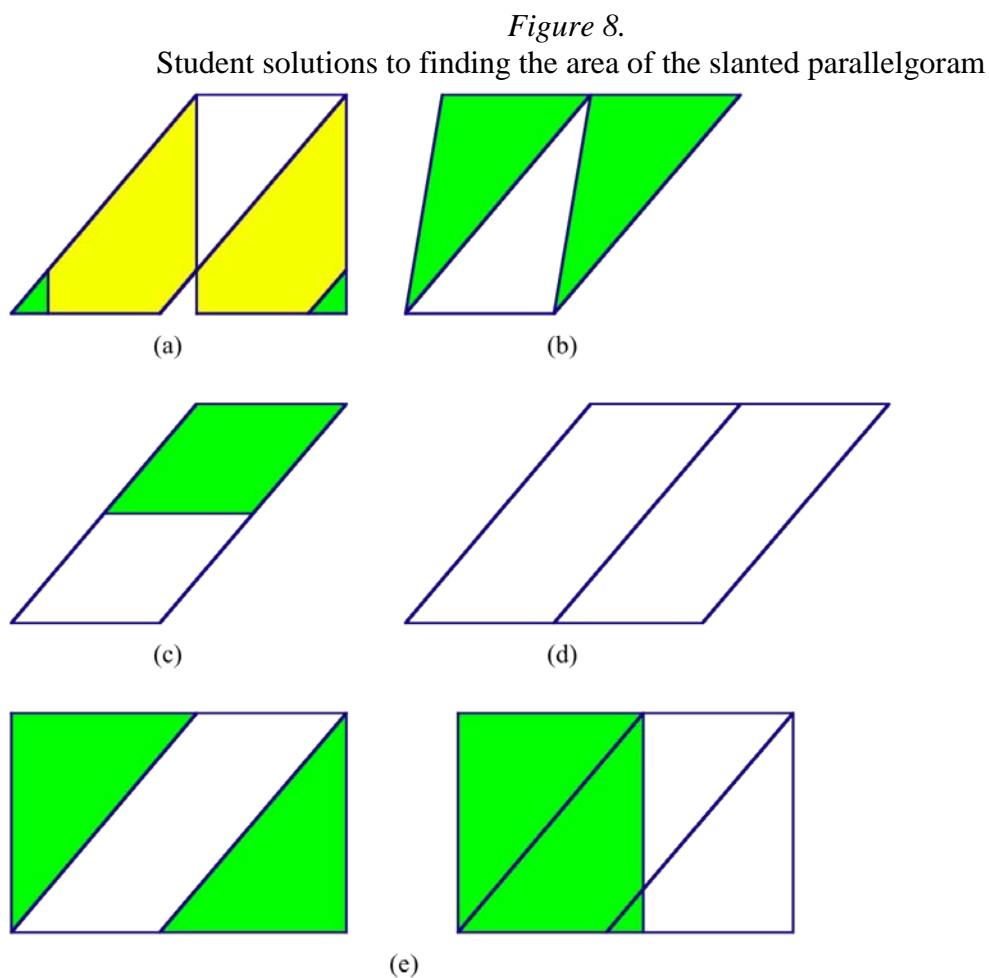
Again, applying the 3 strategies, preservice teachers come up with a variety of ways to calculate this parallelogram. Figure 8 shows some of the typical solution strategies.

Some ways preservice teachers find the area of slanted parallelogram are to use the Cut and Re-Arrange method (see Figure 8 a and b), to employ the Subdivision method (see Figure 8 c), or to use the Make-It-Bigger method (see Figure 8 d and e).

After examining these strategies, students come to the conclusion that the area of a parallelogram is equal to the area of a rectangle built on one of the sides of the parallelogram, and the length of the other side of the rectangle is equal to the distance between the initial side and its opposite side. After defining that dimension of the parallelogram as its *height*, we conclude that the area of parallelograms is the product of the base and the height. Furthermore, they understand that any side of the parallelogram may serve as the base. The class will then move on to examining the area of triangles.

We teach our preservice teachers in the same way we want them to teach their classes. The unique component of instruction "modeled" in this course is the sequencing of topics. It is important for preservice teachers to experience and understand how a mathematical idea may be developed from other mathematical ideas. Therefore, the unit focuses on developing area formulae based on previously established area formulae. The topics are sequenced as follows: (1) area of rectangles and squares, (2) area of L-shape, (3) area of parallelograms, (4) area of triangles, (5) area of other quadrilaterals, and (6) area of circles.

Develop connections between content knowledge and other types of knowledge. As preservice teachers explore the area formulae, they must think as children might, only being able to draw on what has been previously established. Understanding how curricula are developed and how mathematical ideas can be built on previously established mathematical ideas connects content knowledge to knowledge of content and curricula.



Connecting back to the definition of mathematical content knowledge. Although it may appear that the common content knowledge in this example is a specific area formula, we argue that it really is the understanding of how to find the area of an unfamiliar shape (i.e., a shape for which there is no formula yet) by changing it to a familiar shape (or a collection of familiar shapes). Specialized content knowledge is explicitly understanding several different general strategies for creating familiar shapes discussed above. Pedagogical content knowledge is demonstrated by understanding how each of these strategies may be applied to various figures such as parallelograms, triangles, and circles.

Summary/Conclusion

All four teaching scenarios presented in this paper focus on developing mathematical content knowledge for teaching. Each scenario first considers what preservice teachers enter our classrooms with (either as pre-existing conceptions or by explicitly excluding pre-established rules) and builds on that knowledge. Preservice teachers explore and invent methods to investigate a posed problem and then discuss those methods.

In the limited amount of time teacher educators spend with prospective teachers it is impossible to address all of the mathematical topics they may come across in their future teaching. Our main goal, then, is to develop independent, reflective learners that can address new content and new pedagogies as they are presented, and make sense of these new ideas on their own. Still, as instructors of the content courses for the preservice teachers, decisions have to be made as to what content should be taught and how deeply that content should be examined. In our opinion, sufficient time must be allowed to explore selected mathematical ideas in depth, to use multiple representations, and to communicate mathematically. An interesting question then becomes what should be the focus? What is more important, *the* mathematical concept taught or the fact that *a* mathematical concept is *developed* using a “mathematical knowledge needed for teaching” lens?

The authors believe a sole examination of *what the mathematics content should be* misses a more important attention to *how the content should be taught*. Mathematics programs need to encourage preservice teachers to develop a deep understanding of mathematics with a view towards a broader conception of mathematics that incorporates specialized content knowledge, including connections to children’s thinking and the elementary mathematics curriculum.

Acknowledgements: This report is based on continued collaboration and co-presentation of the authors at various conferences over the last four years.

References

- Ambrose, R. C. (1998). A classroom study of invented subtraction strategies (Doctoral dissertation, 1998). *Dissertation Abstracts International*, 59(05), 1497.
- Ball, D. L. (1988). Research on teaching mathematics: making subject matter knowledge part of the equation. In J. Brophy (Ed.) *Advances in research on teaching: Vol. 2. Teachers’ subject matter knowledge and classroom instruction* (pp. 1-48). Greenwich, CT: JAI Press.
- Browning, C. & Garza-Kling, G. (2009). Conceptions of angle: Implications for middle school mathematics and beyond. In Craine, T. & Rubenstein, R. (Eds.) *Understanding Geometry for a Changing World*, 127-140. Reston: National Council of Teachers of Mathematics.
- Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as learning environments for teachers and researchers. In R.B. Davis, C. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics, Journal for Research in Mathematics Education Monograph Number 4* (pp125-146). Reston, VA: National Council of Teachers of Mathematics.
- Conference Board of Mathematical Sciences. (2001). *The mathematical education of teachers*. Washington, DC: Mathematical Association of America.
- Hiebert, J., & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and Instruction*, 14, 251-283.
- Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372-400.
- Hill, H.C. Rowan, B., Ball, K.L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(2), 371-406.
- Hill, H. C., Schilling, S., & Ball, D. L. (2004). Developing measures of teachers’ mathematics knowledge for teaching. *The Elementary School Journal*, 105(1), 11-30.

- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.
- Moss, M. (2006). Specialized understanding of mathematics: A study of prospective elementary teachers. Doctoral dissertation, The University of Tennessee-Knoxville.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Philipp, R. (2005). Integrating mathematics and pedagogy to illustrate children's reasoning. San Diego State University Foundation. Allyn & Bacon.
- Philipp, R. A., Ambrose, R., Lamb, L. C., Sowder, J. T., Schappelle, B. P., Sowder, L., Thanheiser, E., & Chauvot, J. (2007). Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education*, 38, 438-476.
- Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Thanheiser, E. (2009a). *Developing preservice teachers' conceptions of numbers in our base-ten numeration system*. Paper presented at the American Educational Research Association (AERA), San Diego, CA.
- Thanheiser, E. (2009b). Preservice elementary school teachers' conceptions of multidigit whole numbers. *Journal for Research in Mathematics Education*, 40, 251-281
- Wilson, Patricia S. and Verna M. Adams. "A Dynamic Way to Teach Angle and Angle Measure." *Arithmetic Teacher* 39 (January 1992): 6-13.