Secondary-level Student Teachers' Conceptions of Mathematical Proof

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Abstract

Recent reforms in mathematics education have led to an increased emphasis on proof and reasoning in mathematics curricula. The National Council of Teachers of Mathematics highlights the important role that teachers' knowledge and beliefs play in shaping students' understanding of mathematics, their confidence in and outlook on mathematics education, and their ability to use math to solve fundamental problems. It is crucial that teachers, especially the uninitiated, understand on a deep level the mathematical concepts that they are expected to teach to adolescents. Thus, it becomes critical for teacher educators to assess the understanding and abilities of student teachers in constructing mathematical proof. The analysis in this study is based on three factors: 1) meaning of proof, 2) ideas about teaching methods on proof, and 3) ideas about the use-fullness of proof in a mathematics classroom. An analysis of the data collected from this study indicates that current student teachers' conceptions of mathematical proof are limited. The uneasiness expressed by student teachers about mathematical proof may suggest an examination of students' experiences with the mathematical proof in both secondary and post secondary classrooms.

Introduction

The call for reform in mathematics education is as old as the history of school itself. In recent years, the National Council of Teachers of Mathematics (NCTM) has published several documents (NCTM, 1989; 1991; 1995; 2000) to stimulate discussion and debate about reform in mathematics education. In 2000, the NCTM proposed revised content *Standards* with the purpose of making mathematics more meaningful for both teacher and learner. Teachers are now expected to assist their students in meeting these standards. The NCTM documents remind us that learning mathematics requires far more from the student than simply solving exercises by working with symbols, performing desired calculations, and doing routine proofs. Learning mathematics is fundamentally about "developing a mathematical view point", "mathematical reasoning", "communicating mathematically", "making connections" in mathematics, and building "connections" with other disciplines and [among mathematical] experiences (NCTM 2000, 56). Since learning mathematics involves discovery, "proof and reasoning" are powerful ways of developing insights, making connections, and communicating mathematically. NCTM underlines this fact by claiming: "being able to reason is essential to understanding" (p. 56). This suggests that proficiency in "mathematical proof and reasoning" (p. 56) is an integral part of mathematics.

The role of the teacher is critical in this respect. As the NCTM (2000) emphasizes, "[s]tudents learn mathematics through the experiences that teachers provide" (p. 16). The NCTM underscores the fact that teaching shapes students' understanding of mathematics, their ability to use it to solve problems, and their confidence in and attitude towards mathematics. Many researchers have pointed out that teachers' knowledge and belief play a critical role in successfully enacting classroom practices (Fennema & Franke, 1992; Thompson, 1984). Effective teachers understand and truly *know* the mathematics that they are teaching. Furthermore, they are flexible in their teaching practices, drawing on that knowledge appropriately and creatively as they instruct their students. The goal of teacher education programs is to provide the foundation that enables neophyte instructors to grow into effective teachers. This can only be achieved by first understanding the student teacher's conceptions, beliefs, and ability with regards to mathematics. Hence, within the current context of reform, it becomes critical to know more about the student teacher's ability to construct proofs. Student teachers' understanding of and experience with mathematical proof will influence the manner in which they approach teaching this concept. The poise that they demonstrate in teaching mathematical proof, in turn, enhances the soundness of their curricular judgments, their ability to respond to student questions, and their skill in making connections, both within the mathematics curriculum and among other academic disciplines (Grouws & Shultz, 1996).

Research purpose

Jones (1997) notes that the teaching of mathematical proof places significant demands on both the subject matter knowledge and pedagogical knowledge of secondary mathematics teachers. Knuth (2002a; 2002b) also mentions that the teacher's conception of proof influences both the role that s/he assigns to proof in her/his mathematics classrooms and her/his instructional approach in teaching such a concept. The goal of reform efforts is to foster the development of students' understanding and uses of proof. Hence, it is important to examine the prospective secondary teacher's conceptions of proof, for it will be these conceptions and understandings that will ultimately shape classroom practice.

This study examines the degree to which student teachers feel secure in their understanding and conceptions of mathematical proof. Numerous studies related to mathematical proof, undertaken at different levels of schooling and from different perspectives, have been reported in the literature. Studies run the gamut from the university-level perspective of students and teachers (Raman 2003; Housman & Porter, 2003) to the secondary-level point of view of students and teachers (Balacheff, 1988; Healy & Hoyles, 2000; Knuth, 2002a) to the perspective of prospective elementary-level student teachers (Martin & Harel, 1989). With the exception of Jones (1997) and Cyr (2004), little work has been done on secondary-level student teachers' (hereafter mentioned as student teachers) conceptions of proof. I aim to contribute to reducing this gap in the literature: the purpose of this paper is to report on the conceptions of mathematical proof held by student teachers who will be teaching mathematics at the high school level. I take a case-study approach in this research, using written response and interviews as methods for date collection. The study itself involves two-phases: Phase 1: Participants' responses to Written Tasks; and Phase 2: Interview of selected students.

Research Question

The fundamental question that drives this study falls into the category mentioned above: *How do student teachers understand the notion of mathematical proof*? Simply put, the purpose of this study is to examine student teacher's conceptions of mathematical proof. Two primary research questions focus and guide this examination:

1) What are student teachers' conception about the nature and role of proof?
2) What are student teachers' abilities to complete mathematical proofs?

Questionnaire

I developed a questionnaire that would provide the data necessary to answer my research question. The questionnaire consists of two parts. My aim was to assess both student teachers' conceptions of proof and their ability to carry out mathematical tasks. Section (A) of the Questionnaire is designed to provide answers to the question 'What are student teachers' conceptions about the nature and role of proof?' Section (B) of the Questionnaire is designed to provide data that addresses the question, 'What are student teachers' abilities in completing mathematical proofs?'

The 17 student teacher participants in this study were completing the final semester of a teacher education program at a large Canadian university when this study was conducted. Data collection took place two weeks prior to the start of the final practicum (classroom teaching experience). All participants were Mathematics majors who had completed at least twelve 3-credit courses in math. The teacher education program at this University consists of a five-year combined degree. Students are required to take at least twelve 3-credit courses in their subject of major. Students with a Bachelors' degree, were able to enter into an 'after-degree' program typically consisting of two years of additional study in Education. The study was conducted with the approval of University Ethics Committee.

As intended, the data collected did provide information about student teachers' conceptions of, and their proficiency with, mathematical proofs. This paper reports and discusses results from the first part of the questionnaire. I posed three questions in Section A:

a. Describe what the notion of proof means to you.
b. In your opinion, what is the best way to develop students' ability to write proof?
c. In your opinion, after all, is it important for high school students to learn how to write proof? Why?

Analysis

I base my analyses of student teachers' conceptions of proof on three factors: the student teachers' 1) meaning of proof, 2) ideas about teaching methods on proof, and 3) ideas about the use-fullness of proof in a mathematics classroom.

a). Student Teachers' Conceptions about the Meaning of Proof

The idea of proof is pivotal in mathematics. Hence, it is crucial for prospective teachers to have a clear understanding of the meaning of proof. The literature shows that, across the mathematics education community, there is some disagreement as to the meaning of proof. Reid (2002) claims that proof is not well defined, and *cannot be* *defined* as it means different things to different people. The various meanings of 'proof and proving' depend upon the specific and different dimensions within which the concepts are considered. As Reid notes, a definition can be based on the *concept* of proof, the *purpose* of teaching proof, the kinds of *reasoning* involved, or the *needs* that the process of proving is seen to address. All of the student teacher participants in my study thought about the meaning of proof in terms of the role proof plays in mathematics. Even though there is some common ground in their sense of what a proof is, there are also differences.

The table below presents a break down of student teacher responses. One can clearly detect traces of the different meanings of proof in their responses. In this table, I have identified the dominant category relevant to each person's response.

Table 1
Student teachers' dominant meanings of proof

#	Categories	# of student teachers
1	Proof as Verification	9
2	Proof as Derivation	2
3	Proof as Logical Argument	2
4	Proof as Justification	2
5	Proof as Discovery	1
6	Proof as Explanation	1

1. Proof as Verification

Hanna (1983) argues that the main role of proof in mathematics is to demonstrate the correctness of a result or truth of a statement. From the earliest beginnings of mathematical proof, during the time of the ancient Greeks, the fundamental role of proof has been to verify mathematical results. However, there is a downside to this argument—it ignores the results derived by using non-deductive means. Proof as verification is often associated with formal proof and mathematicians' proof, which is often considered 'real proof,' proof that is rigorous and certain. The mathematician uses this mode of proof to present his/her new results to the mathematics community, understanding that colleagues will necessarily verify the result. If the only role played by proof was the verification of results, one could argue that including proofs in the school curriculum would be of little use since the literature demonstrates that students are more convinced by examples than they are by rigorous/formal proofs (Harel & Sowder, 1998). Formal proofs may seem less convincing to students (or to those who are not mathematicians) because of their symbolic nature and operators.

A majority of the student teachers thought that the meaning of proof is something that can be used to verify an algorithm or a statement. To quote some of the written responses,

> "Proof means that you have shown the relationship you are exploring is true, no matter what." (George)
>
> "Verifying that an algorithm or mathematical statement is correct." (Clare)
>
> "A proof is showing that the concept is true for <u>all</u> cases." (Chandelle) (emphasis in the original)

It is not surprising that more than half of these student teachers understood proof as 'something used for verification' since this is its most commonly used dimension of meaning. For over three decades, the literature (Hanna, 1983) has identified the main role of proof in mathematics as verification. Indeed, Reid (1995; 2005) calls this the traditional concept of proof precisely because it goes back far in time. In most classrooms, proofs are for 'verification' of some already known mathematical results.

2. Proof as Derivation

Two student teachers explicitly used the terms *"derivation"* and *"derive"* in the definitions each provided for proof. For example,

> "detailed (though not necessary) derivation." (Tahira)

> "a proof is a system of equations/relationships that are used to derive another equation /relationship from a previous understanding." (Brandon)

What these two participants probably had in mind was the derivation of the quadratic equation as suggested by Knuth (2002a). They might have used the term to show that proof is something that demands the requirements of abstraction, formal vocabulary, symbols, founded on a set of formal axioms etc., that satisfies the requirements of a professional mathematician (Hersh 1993; Marrades and Gutierrez 2000).

3. Proof as Logical Argument

For research mathematicians, a proof of a statement is synonymous with the logical reasoning that makes the statement true (Lucast, 2003). For this function of proof, all that is needed is a series of arguments that are logical. This was evident in the responses below:

> "A proof is a solution to why a theorem is true." (Deanna)

> "A statement, based on logic, that indisputably "proves" a theorem to be true or un-true, so long as there is no fault in underlying logic or assumption." (Philip)

4. Proof as Justification

When we think about proof, one of the first things that comes to mind is its role in ensuring the truth of some fact. Proof demonstrates to us the truth of intuitively correct facts (Lucast, 2003). The following responses suggest an understanding of proof as justification:

> "A mathematical justification of a statement using previously proven facts." (Wendy)

> "The mathematical process that justifies a formula or fact." (Grace)

Yet, since most students find examples more convincing than rigorous proofs (Coe & Ruthven, 1994), it would seem that examples, more so than proofs, provide *justification*. Rodd (2000) noted that most of the time, *justification* is used in a non-technical way to mean 'a rationale for a belief'. Hence, understanding proof solely in terms of justification may limit one's ability to work with proof.

5. Proof as Discovery

Proof plays an important role in the discovery or creation of new mathematics (Knuth, 2002a). Indeed, one may argue that the 'discovery function' of proof is especially important in the mathematics classrooms. The NCTM (2000), for example stresses the importance of using discovery to learn mathematics. Interestingly, according to Chazan and Yerushalmy (1998), proof plays a greater role as discovery in classrooms where software is used. The research data, unfortunately, suggest that student teachers do not commonly understand proof in terms of discovery. Only 1 of 17 student teachers recognized that proof could also be used in the discovery or creation of new mathematics:

> "Discover of a concept, formula, identity, theory." (Gita)

6. Proof as Explanation

In school mathematics, the main purpose of introducing proof is to enhance understanding. Hence, the explanatory value of proof is of great importance in the school context (Leddy, 2001; Hanna, 1990). Yet, only one student alluded to this function. The participant Spencer writes that ". . . You can do a proof for a law, algorithm or a relationship[. . .] It is meant to be an explanation of why this law, algorithm or relationship is the way it is." I find it extraordinary that only 1 student teacher out of 17 considered this meaning of proof. However, this probably reflects the reality within school classrooms. Teachers probably do not place much emphasis, or any at all, upon proofs that explain.

None of the student teachers saw proof as a means to *promote* understanding. As Knuth (2002a, 2002b) notes, students are rarely able to identify the main objectives or functions of mathematical proof, likely because they are only exposed to proof within the realm of geometry as the verification of an already known result. My study validates his finding.

b). Student Teachers' Conceptions about the best way to develop students' ability to write proof

Student responses can be categorized in the following manner:

Table 2
Student teachers' suggestions about the way to write up a proof

#	Categories	# of student teachers
1	Step by Step Procedure (or teacher-assisted examples)	13
2	Constructivist	3
3	Social Process	1

A considerable majority of the students thought that the best way to teach proof is through teacher-assisted examples and step-by-step procedures. Some of the responses include the following:

> "Practice, show examples of proofs, give steps for determining what direction to take the proof in." (Grace)

> "Assisted stage by stage implementation and simple individual proofs." (Cathy)

> "Give a few examples of a proof and explain. Give them the structure of a proof. Then give them a very simple proof they can do on their own so they can get the feel of them." "Going through numerous examples in class." (John)

> “The best way to develop students’ ability to write proof is to start with easy proofs that they understand.” (Daniel)

> “First develop a good understanding of what constitutes a proof. Second – understand some common strategies like: Third – practice deductive reasoning . Fourth – practice proving identities using axioms.” (Brian)
> “Exposure to some basic proofs; Description of step by step process of proofs. Practice, Practice, Practice.” (Brandon)

> “Have them practice with simple versions then work their way to more complicated questions.” (Wendy)

> “Practice. Teach them how to reason.” (Gita)

> “To introduce the idea early using simple examples. Challenge students to ask how they know something is true, rather than always just assuming it is. Ask students to proceed from one step in the logic to the next” (Philip)

In all of these responses, the only method for learning proof that students were able to suggest is ‘more practice’. This likely reflects the way in which they were taught math. Many of these student teachers may have experienced mathematics in a classroom where the teacher and the text were authorities for the right answer. In traditional mathematics classrooms, teachers, and therefore students, view mathematics in terms of procedures to be memorized; the emphasis in such a context is on mechanistic answer-finding. Students who have had such experiences often believe that the only way of mastering the subject is to treat it as a body of isolated concepts and procedures that must be memorized.

Yet, when student teachers are given an opportunity to reflect on possible teaching methods, their responses suggest an unconscious admission that there is something wrong with this age-old and familiar approach. I find this encouraging. Unfortunately, they are unsure as to what other methods to use, given that they probably received little exposure to the *other methods* while they were students in grade school classrooms. This is reflected in Deanna’s suggestion that

> “Students should be exposed to different ways that theorems can be proved, so they realize there is more than one way to write a proof. In this sense students will not be anxious or nervous when they need to prove something.” (Deanna)

Some of the students were familiar with the word ‘Constructivism’ and the phrase ‘Constructivist approach,’ though they were often quite vague in their use of these terms. A few of them were not sure what “constructivist” teaching entailed:

> “Using constructivism, guiding them to discover idea.” (Gita)

Clare’s idea of a constructivist approach is a

> “Step by step procedure
> (1) Write what you know
> (2) What are you trying to show
> (3) What do you know that can help simplify or make the statement easier and so on ….”

Indeed, she seemed unaware of the limits of her understanding, for she specifically used the term “constructivist approach” in her written response and offered verbally, to her peers in the classroom, her version of “constructivism.”

Most of the student teachers did not commonly conceive of mathematics as something that arises from the collective energy of groups of individuals. Only one participant considered proof as arising from or as a product of social inter-action: “I think the best way is to get students to come up with a proof as a group” (Sara). Sara’s perspective has something in common with the complexivist view. Complexivists suggest that learning takes place, not within the individual mind, but within “an ongoing *structural dance*—a complex choreography—of events which, even in retrospect, cannot be fully disentangled and understood, let alone reproduced” (emphasis in the original; Davis et al. 1996, 153). By framing the classroom as a “complex, emergent system, as an individual collective learner rather than as a collection of individual learners” (Davis and Upitis 2004, 126), teachers are able to observe the “thinking” of the collective, . . . [that is,] the interactions and prompts that trigger new possibilites and insights for the collective” (Davis and Simmt 2003, 144).

It is to be noted that only one of 17 students stressed co-construction of knowledge. Although social acceptability is an important aspect of mathematical proof—as Manin (1977) writes, “a proof becomes a proof after the social act of accepting it as a proof” (as cited by Knuth, 2002b, p. 64)—co-creation of proof through social problem solving seems not to be something that these student teachers have deeply considered.

c). Student Teachers’ Conceptions about the importance of learning proof in high school

Student responses tend to indicate three lines of thinking. A few students responded to the question “Is it important to learn proof in high school?” with an emphatic “No.” A second group of students, also low in number, said that proof should be taught in every mathematics class and then offered reasons for this belief. A third group of students believed that proof should be part of the curriculum; however, they also expressed reservations about emphasizing it. Most of the student teachers in this group also mentioned that mathematical proof should only be introduced to select groups of students, including those who plan to study advanced mathematics. As Hanna (2000) noted the idea that proof should be reserved for a ‘certain group of students’ is highly problematic. This attitude implies that it is possible to learn mathematics, without ever needing to prove a theorem. Hanna (2000) writes that:

> The basis for this idea seems to be the erroneous assumption that proof is, in fact, a specialized branch of mathematics, even an arcane branch so complicated that it is best avoided by all who do not absolutely require it. […] Reserving proof for those planning post secondary studies cannot but send the message to the bulk pf the students that for them there is really no point in proof at all. (p. 25).

The following are some of the student teachers’ responses that reflect this negative attitude to proof:

> “…..formal proofs are very complicated and mostly confusing and frustrating to high school students.” (George)

> “in geometry proof is a must, but trigonometric proof[s] are above most students.” (Sara)

> "Not formal proofs, but proofs that explain." (Spencer)
>
> "Only students going to university should be taught proof." (Daniel)
>
> "It should only be used in pure stream." (Gita)
>
> "No unless they are planning to take advanced mathematical courses." (Wendy)
>
> " Introduce proof – let them not be asked to write. High school topics are difficult even without proof." (John)
>
> "No – it is far too complex." (Tahira)
>
> "It is important for verification purpose. They [students] should not necessarily be able to come up [with proof] on their own." (Chandelle)

Overall, these students consider 'mathematical proof' a waste of time in the secondary-level classroom. Most participants regard proof as something written in two columns by the teacher on a black/white board for students to copy and memorize for an exam. They also strongly believe that adolescents are not likely to use mathematical proof anywhere outside of the classroom: As Terrence stated: "No, as students will not use proof again [after their high school]."

Also of interest are the responses of the student teachers who want to retain proof as a part of the high school curriculum. What is especially intriguing is their ambiguity. They do not wholeheartedly endorse the teaching of proofs at the high school level (mostly because of their discomfort with them), but they argue that inclusion will improve fluency in problem solving, and for this reason, "proofs should be retained." They are also aware of the fact that, in post-secondary education, mathematical proof is a must, and a lack of ability in completing mathematical proofs may cost students, especially if their major is in mathematics or a related subject area. George mentioned that "[i]t is important to teach our students to be able to logically discern information much like proof"; Clare said students should be introduced to"[v]ery basic proof such as trigonometric proofs . . . otherwise no"; and Sara thought that trigonometric proofs "are above most students." By trigonometric proofs, Clare probably meant trigonometric identities. She continues: "I feel students should be able to reassure themselves that what they are doing will lead to a correct result […] I think proofs allow practice with problem solving, except only with the basic proof." Additional responses of note are listed below:

> "Yes students should be able to write proofs so that it will aid in developing their understanding." (Deanna)
>
> "It [proof] helps not to memorize a theory or concept." (Gita)
>
> "proofs should be taught as it increases confidence in mathematics." (Gita)
>
> "students going to university should be taught proof." (Daniel)
>
> "Important as it develops helps in deductive reasoning skills which can be applied to every day

logic problems. ….gives insight to how an algorithm was developed"
(Daniel)

"Yes- because students should not look at some pattern and believe the pattern is universal."
(Philip)

"Yess [*sic*], helps understanding prepares them for post secondary school. Helps in deeper thinking. Helps to make connection." (Grace)

Finally, some student teachers suggested that proof should be taught in high school so that the method of proof learned in geometry could be transferred to other contexts. Reid (2005) disagrees with this idea, arguing that the criterion of acceptability of explanation that defines mathematics as a domain of explanation cannot be transferred to other domains. He suggests that the motivation for teaching/learning proof must be a better understanding of the nature of mathematics itself, not better reasoning in other domains.

The responses of the student teachers' in this study substantiate Knuth's (2002a) observation that the teaching of proof is restricted to certain areas and topics in the secondary school curriculum. In the secondary school curriculum, proof is most commonly relegated to the role of *verification*. Since proofs are, therefore, largely about *verifying* an already known mathematical result, students find proof of little use in school mathematics. Responses provided by respondents in this study suggest that these prospective teachers are unsure at to what would constitute an effective strategy for learning and/or teaching mathematical proofs. One of the complications associated with teaching proof is the difficulty of communicating an effective strategy for proving (Hadas et al. 2000). Training student teachers to make proof accessible to high school students should be one of the focuses in teacher education. Knuth (2002c) had noted that the teachers should experience proof as a learning tool. University educators should seriously think about this if they hope to make mathematics more meaningful for both teacher and learner.

Summary

My analysis of the data collected indicates that student teachers' conceptions of mathematical proof are narrowly grounded in traditional understandings of its nature, function, and value. Most of the participants in my study regarded proof as a means of verification; only one person understood proof as a means of explanation. My findings, though they pertain to student teachers, support Knuth's (2002a, 2002b) conclusions concerning practicing teachers. Most student teachers believe that teacher-assisted practice is the best way to learn proof. Although some students referred to "constructivist approaches," most were unclear about what a "constructivist approach" entails. My sense is that students included the term in their responses to show that they were aware of current educational trends.

Many student teachers also believe that proof should only be introduced to select groups of students such as those who plan to study advanced mathematics. The fact that student teachers today would take such a position raises a red flag, for it sends the message that there is no need for the majority of students to study proof and proving in the classroom, and so little need for future teachers to teach proof to all of their future students. For the most part, their responses suggest that these student teachers continue to regard proof as a formal, and often meaningless, exercise performed by the teacher. (Interestingly, Alibert [1988] conducted a study twenty years earlier with a similar result: those participants also regarded proof as a formal, and often meaningless, exercise performed by the teacher.) The student teachers I interviewed also spoke with some frustration about their experiences with proof and proving.

The uneasiness expressed by student teachers about mathematical proof suggests an examination of students' experiences with the mathematical proof in the post secondary classrooms too. Wu (1997) suggests that the undergraduate mathematics education is built on "Intellectual –Trickle Down Theory" (p. 5). Professors direct their teaching towards the best students while believing that, somehow, the rest will take care of themselves. He also notes another aspect of undergraduate mathematics education, what he calls *"delayed gratification"* , where the instructors believe that if students don't understand something in a particular course now, they will surely come to understand it when they do their graduate study or when they begin research. However, in reality only 20% or less of the math graduates continue with graduate work. For the vast majority of undergraduates, these courses are the "grand finale of their mathematical experience" (1997. 4). Those who opt to become secondary mathematics teachers will be among this vast majority of students who were fed "technicality after technicality" in their mathematics courses, and did not understand most of the material that was taught.
My study points towards a need for further research in this area. Further in-depth studies of student teachers' understanding of mathematical content areas other than "mathematical proof" are necessary if teacher education programs are to become more effective at educating future mathematics teachers.

References

Alibert, D (1988). Towards new customs in the classroom. *For the Learning of Mathematics* 8 (2), 31-35.

In Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.) *Mathematics, teachers and children,* (pp. 216-235). (Trans. By D. Pimm). London: Hodder & Stoughton.

Coe, R ., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal, 20,* 41-53.

Chazan, D., & Yrushalmy, M (1989). Charting a course for secondary geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space,* 67-90. Mahwah: Erlbaum.

Cyr, S (2004). Conceptions of proof among pre-service high school mathematics teachers. In Proceedings of the 26th Annual meeting of North American Chapter of the International Group for the Psychology of Mathematics Education, D. E. McDougall and J.A. Ross (Eds). 2, 569-574. Toronto, Canada: University of Toronto.

Davis, B., Sumara,D.J, & Kieran, T.E (1996). Cognition, co-emergence, curriculum. *Journal of Curriculum Studies*, 28, 151-169.

Davis, B., and E. Simmt. 2003. Understanding learning systems: mathematics education and complexity science. *Journal for Research in Mathematics Education*, 34(2): 137-167.

Davis, B., & Upitis, R. (2004). Pending knowledge: on the complexities of teaching and learning. *Journal of Curriculum Theorizing* 20(3), 113-128.

Fennema, E., & Franke, M (1992). Teachers’ knowledge and its impact. In D. Grouws (ed.) *Handbook of Research on Mathematics Teaching and Learning*, 147-164. New York: Macmillan.

Hanna, G (1983). *Rigorous proof in mathematics education.* Toronto: OISE Press.

Hanna, G (1990). Some pedagogical aspects of proof. *Interchange* 21(1), 6-13.

Hanna, G (2000). A critical examination of three factors in the decline of proof. *Interchange* 31(1), 21-33.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education* 31(4), 396-428.

Hersh, R (1993). Proving is convincing and explaining. *Educational Studies in Mathematics* 24(4), 389-399.

Housman, D., & Potter, M. (2003). Proof schemes and learning strategies of above average mathematics students. *Educational Studies in Mathematics* 52(3), 139-158.

Hadas, N., Hershkowitz,R., & Schwarz, B.B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics*, 44(1-2), 127-150.

Jones, K. (1997). Student teachers’ conceptions of mathematical proof. *Mathematical Education Review*, 9, 23-32.

Knuth, E.J. (2002a). Secondary school mathematics teachers’ conceptions of proof. *Journal for Research in Mathematics Education* 33, no. 5: 379-405.

__________2002b. Teachers’ conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, 5, 61-88.

____________2002c. Proof as a learning tool. *Mathematics Teacher* 95 (7): 486-490.

Leddy, J.F.J. (2001). *Justifying and proving in secondary school mathematics.* Unpublished doctoral dissertation, Toronto, Canada: OISE, University of Toronto.

Lucast, E.K. (2003). *Proof as method: A new case for proof in mathematics curricula.* Unpublished masters' thesis. Pittsburg, US: Carnegie Mellon University.

Martin, W.G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal forResearch in Mathematics Education*, 20 (1), 41-51.

Marrades, R., & Guiterrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics*, 44, 87-125.

Manin, Y. 1997. *A course in mathematical logic.* New York: Springer Verlag.

National Council of Teachers of Mathematics (1989). *The Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

_________________ 1991. *Professional standards for teaching mathematics.* Reston, VA: National Council of Teachers of Mathematics.

_________________1995. *Assessment standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

__________________. 2000. *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Raman, M. (2003). Key ideas: what are they and how can they help us understand how people view proof? *Educational Studies in Mathematics* 52 (3), 319-325.

Rodd, M. M. (2000). On mathematical warrants: Proof does not always warrant, and a warrant may be other than proof. *Mathematical Thinking and Learning* 2(3), 221-244.

Reid, D. (1995). The need to prove. Unpublished doctoral dissertation, Edmonton, Canada: University of Alberta.

Reid, D. (2002). What is proof? *La Lettre de la Preuve: International newsletter on the teaching and learning of proof.* Summer 2002. Online at: http://www-leibniz.imag.fr/DIDACTIQUE/preuve/Newsletter/02Ete/WhatIsProof.pdf

Reid, D. (2005). *The meaning of proof in mathematics education* Paper presented to Working Group 4: Argumentation and Proof, at the Fourth annual conference of the European Society for Research in Mathematics Education. Sant Feliu de Guíxols, Spain. 17 - 21 February 2005. Proceedings to appear. Currently available online at: http://cerme4.crm.es/Papers%20definitius/4/Reid.pdf .

Thompson, A.G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics* 15 (2), 105-127.

Wu, H. (1997). On the education of mathematics teachers. Retrieved March, 23, 2007 from math.berkeley.edu/~wu/teacher-education.pdf.