A Lesson Based on Student-Generated Ideas: A Practical Example Highlighting the Role of a Teacher

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Abstract

The role of a teacher is different from that in traditional mathematics instruction when the implementation of a lesson is based on students' ideas. The author's experience teaching the same lesson (of the latter format) to two different classes of pre-service teachers in an elementary mathematics methods course is described. Since whole-class discussions were conducted using student presentations, the two classes "looked" very different, emphasizing the unpredictability of the teaching method. The practical example serves as a context to discuss the implications for the role of a teacher when preparing for and conducting lessons centered on students' approaches to problems.

Introduction

The only thing that is predictable in teaching is that classroom activities will not go as predicted (Simon, 1995, p. 133).

A common mathematics lesson format starts with a lecture portion introducing new content followed by guided practice and then independent practice. The teacher models the strategy that she wants the students to use and then they are given the opportunity to practice that prescribed approach on a set of problems. Now imagine an alternative. Suppose a teacher lets the students figure out how to solve a problem in their own ways. After giving students ample time to explore a task, the teacher then guides a whole-class discussion during which the students share their ideas with each other. In preparing, a teacher can anticipate how students might approach a task, what troubles the students might encounter, how the discussion might progress, and what questions she can ask. However, until the students start working, she cannot be certain. Although all lessons have an element of unpredictability, as indicated by the introductory quote, unexpected ways in which students think about problems are more likely in the latter format. After a theoretical perspective is established, the execution of a lesson (of the second format) in two different classes of pre-service elementary teachers in a mathematics methods course is described. The classes were taught back-to-back, highlighting the differences between the two and the unpredictability of the instructional method. To conclude, implications for teachers who use this teaching approach are discussed.

Background

With traditional mathematics instruction, the lesson format initially described, the teacher is viewed as the disseminator of information. Freire (1989) portrays this view of education as "banking education" in which "the teacher is the Subject of the learning process, while pupils are mere objects" (p. 59). The teacher, the authority in the classroom, transmits knowledge to the students who passively receive the information. In doing so, the instructor makes the decisions about the "best" way for students to think about a problem.

In contrast to banking education, Freire (1989) offers an alternative, problem-posing education, in which the role of the teacher is re-conceptualized. "The teacher is no longer merely the-one-who-teaches, but one who is himself taught in dialogue with the students, who in turn while being taught also teach" (Freire, 1989, p. 67). The locus of control is shifted; the students' ideas and comprehension become central. Hiebert et al. (1997) similarly promote teaching and learning with understanding and define understanding in the following manner: "We understand something if we see how it is related or connected to other things we know" (p. 4). Hiebert et al. also describe components of a classroom that promote understanding one of which is the role of the teacher.

> Instead of acting as the main source of mathematical information and the evaluator of correctness, the teacher now has the role of selecting and posing appropriate sequences of problems as opportunities for learning, sharing information when it is essential for tackling problems, and facilitating the establishment of a classroom culture in which pupils work on novel problems individually and interactively, and discuss and reflect on their answers and methods. The teacher relies on the reflective and conversational problem solving activities of the students to drive their learning. (p. 8)

In other words, the major responsibilities of a teacher are choosing and creating tasks and developing a culture within the classroom in which reflection and communication are predominant.

Developing a classroom culture, as described by Hiebert et al. (1997), is a continual process. The culture, or social norms, of the classroom are mutually negotiated between the teacher and the students, especially when the students are accustomed to traditional mathematics instruction (Cobb, Yackel, & Wood, 1993). Hiebert et al. identify two responsibilities for the teacher in establishing the classroom culture: highlight different methods and redefine the role of the teacher. The students need to learn the type of mathematical activity that is valued. Instead of following a fixed procedure with a focus on getting the correct answer to a problem, the different methods used by students become the center of classroom interaction. In this situation, the teacher is no longer the sole authority on correctness. As a community, all members of the classroom have the responsibility to listen to each others' approaches and compare and evaluate them based on the logic of the mathematics. The teacher needs to guide the classroom discourse about the students' explanations of methods.

The teacher also chooses and/or creates a sequence of tasks that will support the students in meeting particular learning goals. "This means that the selection of appropriate tasks includes thinking about how tasks are related, how they can be chained together to increase the opportunity for students to gradually construct their understanding" (Hiebert et al., 1997, p. 31). To pick the activities, a teachers needs to understand the critical mathematical concepts and the ways in which students think about those ideas, knowledge which is accrued through experience working with students as well as by accessing the appropriate resources (Hiebert et al.). Simon (1995) coined the term *hypothetical learning trajectory* to describe this process of identifying the learning objectives, selecting the tasks, and hypothesizing how the students' ideas and comprehension will develop through engaging in the chosen activities. Since the students' thinking and understanding are only predicted, the hypothetical learning trajectory is continually reassessed as the teacher observes and interacts with the students while they are solving the problems.

We used these guiding principles to develop our elementary mathematics methods course and to create the lesson that is described in this article. Based on the various methods employed by the students in the two different classes, the hypothetical learning trajectory was modified differently in each class.

The Course and Students

The lesson that is described occurred in the second semester of a two-semester mathematics methods course for elementary pre-service teachers. In this course, the mathematics content, including number and operations, geometry, measurement, data analysis, and probability, and methods to teach this content were integrated.We taught the concepts by modeling strategies that could be used in the elementary classroom. The students were engaged in doing mathematics throughout the class time.

There were 20 students in one section of the course and 19 in the other. In general, the students in this course were familiar with the mathematical procedures, but often lacked an understanding of the conceptual underpinnings of those procedures. We asked the students to engage in exploratory activities which exposed them to multiple invented strategies and algorithms with an emphasis on the logic and reasoning behind them. They shared their thought processes in small groups and whole-class discussions. For many of the students, this was a new approach to learning mathematics and some expressed frustration with the cognitive disequilibrium that was created. For example, during the first semester, one student said, "This is not like other mathematics classes. You make us think." By the second semester, the students were more accustomed to my expectations and the learning process; a classroom culture with a focus on learning for understanding was developing.

The Lesson Plan

We opened the sequence of two lessons by giving the students a writing prompt: How did you learn to divide fractions? The students then worked though two sets of division of fraction problems. By working through the first set, we intended for the students to discover the common denominator algorithm. We chose the second set to help the students develop an argument to explain why the invert-and-multiply algorithm can be used to divide fractions. In the class described, the pre-service teachers worked through the first set of problems:

a. $6 \div \frac{3}{4}$

b. $2\frac{1}{4} \div \frac{3}{4}$

c. $4\frac{2}{3} \div 1\frac{1}{6}$

d. $2\frac{1}{4} \div \frac{2}{3}$

e. $2\frac{1}{2} \div 3\frac{1}{3}$

The instructions specifically indicated that the students were to complete the problems *without* using an algorithm. Instead of practicing a procedure with which the students were familiar, my objective was for the students to use invented strategies, be exposed to an algorithm less commonly used, and develop an understanding of the more commonly used invert-and-multiply algorithm. The pre-service teachers were also given the option of using any available manipulatives such as counters, pattern blocks, Cuisenaire rods, or fraction strips. We deliberately chose and ordered the five problems. They increased in difficulty providing the students with the opportunity to construct their understanding. The quotient of the first three problems was a whole number and the quotient for the last two problems was a mixed number and proper fraction, respectively. The dividend and the divisor in the second problem had a common denominator; whereas, the dividend and divisor in the last three problems did not.

After the class had the opportunity to work through the five problems in small groups, several students presented their strategies using a document camera. From our classroom observations, We attempted to choose a sequence of approaches that built towards the objective of my lesson, in this case, the discovery of the common denominator algorithm for dividing fractions.

The Implementation of the Lesson

In both classes, the students' responses to the writing prompt were essentially different versions of the saying, "Invert and multiply. Don't ask why." We followed up by discussing the two messages that this maxim implies. First, a procedural orientation is supported by giving steps to follow to divide fractions. Second, students' curiosity about why a procedure works is potentially stifled. Although the students spent a significant portion of the class time working on the problems in groups, the focus of this article is on the subsequent whole-class discussions. The structure and flow of the conversation varied significantly between the two classes since it was based on the strategies used by the students. The following descriptions of what happened in each class are based on fieldnotes and student work.

The First Class. The group that shared their strategy for the first problem was the only group that used a physical manipulative (colored chips). The students represented the dividend using 24 chips with four chips per unit. Three-fourths of a unit was signified by one color and the remaining $\frac{1}{4}$ with a different color. Initially, they counted the six sets of $\frac{3}{4}$ and then realized that there was an additional two sets of $\frac{3}{4}$ for a total of eight (Figure 1).

For the second problem, the presenters used circles (divided into four equal sections) to denote the unit and marked off sets of $\frac{3}{4}$ to find the quotient (Figure 2).We asked the class to compare the two approaches. The students noted that a unit is represented differently (sets of objects versus whole objects). Despite this distinction, each unit consisted of four equal parts. Both groups also used a measurement model of division; that is, they were figuring out how many sets of the divisor made up the dividend. For example, the students determined that there are three sets of $\frac{3}{4}$ in $2\frac{1}{4}$.

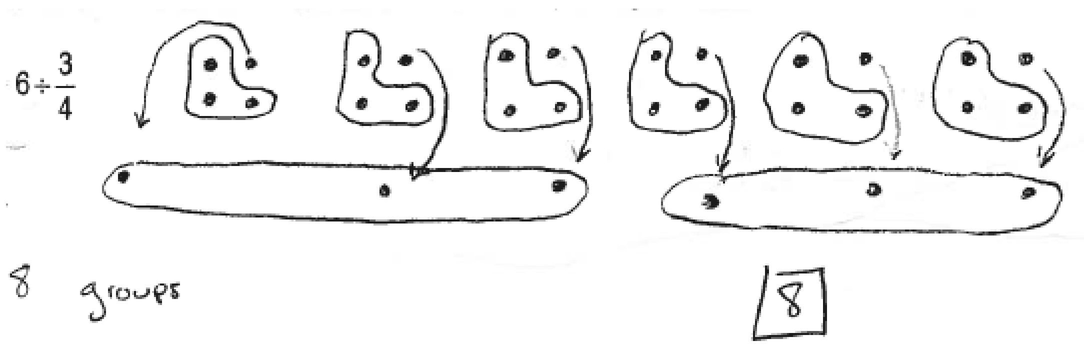

Figure 1.
Strategy using colored chips for the first problem in the first class.

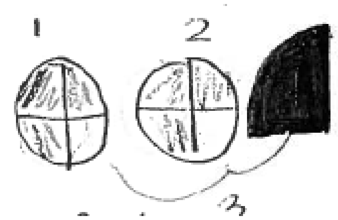

Figure 2.
Strategy using circles as unit for the second problem in the first class.

Similar to the students that used circles, the group that presented the third problem used a rectangle to represent each unit in the dividend (Figure 3), dividing each unit into sixths to determine the number of $1\frac{1}{6}$'s that made up $4\frac{2}{3}$. When we asked how this problem was different from the previous two problems, the pre-service teachers explained that they needed to get a common denominator. (It is important to note the students introduced the idea of common denominator into the conversation.) We asked the class to consider this statement by examining the previous two problems. The students were also finding a common denominator for the first problem when they used 24 chips.

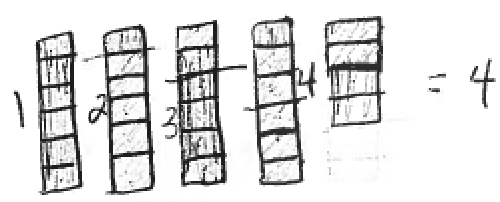

Figure 3.
Strategy using rectangles as unit for the third problem in the first class. (This is not the actual work presented but the work of another student that used the same model.)

One group explained its strategy, also using rectangles, for the last two problems. For the fourth problem (Figure 4), we followed up by asking why the quotient was expressed in eighths rather than twelfths, the common denominator. Another student pointed out that her group had posed this question and figured out an answer: Since they needed to figure out how many $\frac{8}{12}$'s made up the $2\frac{1}{4}$, they were counting sets of eight. We discussed how there is a transition from the original unit when representing the dividend and the “new unit” when using the divisor to determine the quotient.

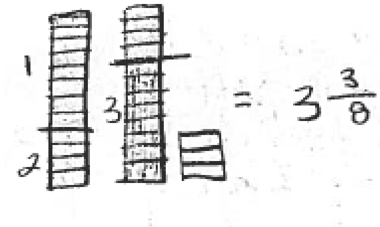

Figure 4.
Strategy using rectangles as unit for the fourth problem in the first class.

For the last problem (Figure 5), we asked the class about the difference between this problem and the previous four. The students noted that the dividend was smaller than the divisor. When we inquired about how this information affected the quotient, the pre-service teachers responded that it was less than one. (Initially, several students said that the quotient would be a fraction. We encouraged these students to be more specific.)

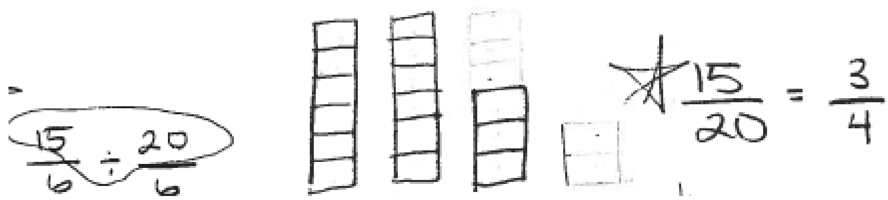

Figure 5.
Strategy using rectangles as unit for the fifth problem in the first class.

On the handout displayed on the document camera, the presenter had written $\frac{15}{6} \div \frac{20}{6}$ and the quotient $\frac{15}{20}$ (Figure 5). We pointed out these two expressions and the students realized that the answer was the quotient of the two numerators when the dividend and divisor were expressed with a common denominator. Since the pre-service teachers were curious about whether this strategy would always work when dividing fractions, we suggested that they try using it for the previous problem. (It is essential to mention to students that this process does not constitute a proof.) Once they verified the strategy yielded the quotient for the fourth problem, we gave the students the task of writing the steps for this algorithm (Figure 6).

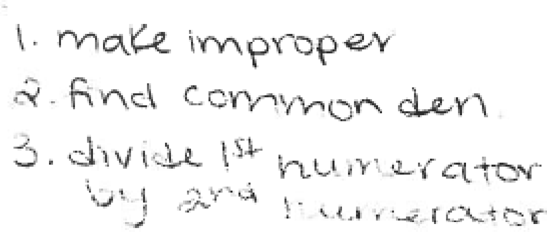

Figure 6.
The steps of the common denominator algorithm written by a student in the first class.

We also made a connection between the common denominator algorithm and the invert-and-multiply algorithm. In the case when the denominators are the same, they divided to one after the divisor is inverted.

$$\frac{15}{6} \div \frac{20}{6} = \frac{15}{6} \times \frac{6}{20} = \frac{15}{20}$$

The Second Class. The whole-class discussion in the second class started in a similar manner as the first. The group that presented the first problem used the colored chips concluding that there were eight groups of $\frac{3}{4}$ as labeled in Figure 7.

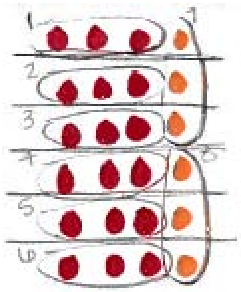

Figure 7.
Strategy using colored chips for the first problem in the second class.

At this point, the conversation began to diverge from the previous class. Two of the groups used repeated addition to determine the quotient. One of these groups presented this strategy for the second problem (Figure 8). The students added $\frac{3}{4}$ and $\frac{3}{4}$ to get $1\frac{1}{2}$ which they then converted to $\frac{6}{4}$. Adding a third $\frac{3}{4}$ resulted in the dividend of $\frac{9}{4}$ which enabled the pre-service teachers to conclude that the quotient was three. They also wrote the dividend as an improper fraction and realized that, with a common denominator, they could simply divide the numerators to determine the quotient. For the remainder of the problems, this group used the common denominator algorithm.

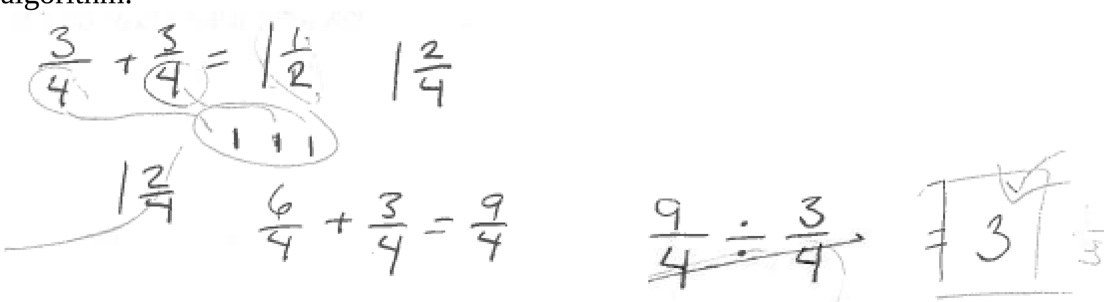

Figure 8.
Strategy using repeated addition for the second problem in the second class.

The student who presented the third problem thought about it from a partition perspective of division, addressing the question: How much is one unit? For example, she changed the division problem to an equation: $4\frac{2}{3} = 1\frac{1}{6} \times y$ (Figure 9). The *y* represents the unit; that is, $1\frac{1}{6}$ of the unit is $4\frac{2}{3}$. She needed to figure out what part of the $4\frac{2}{3}$ represented $\frac{1}{6}$. Since there were seven $\frac{1}{6}$'s, she divided the $4\frac{2}{3}$ into seven equal pieces and removed one of those pieces to find the unit or the quotient which was four. We had debated whether to ask this student to present her strategy. When we visited her group, we listened to her help her classmates and initially had trouble understanding her approach, one that we had not anticipated the students to use. In addition, the method showed evidence of the invert-and-multiply algorithm rather than the common denominator algorithm, the objective for this particular lesson.

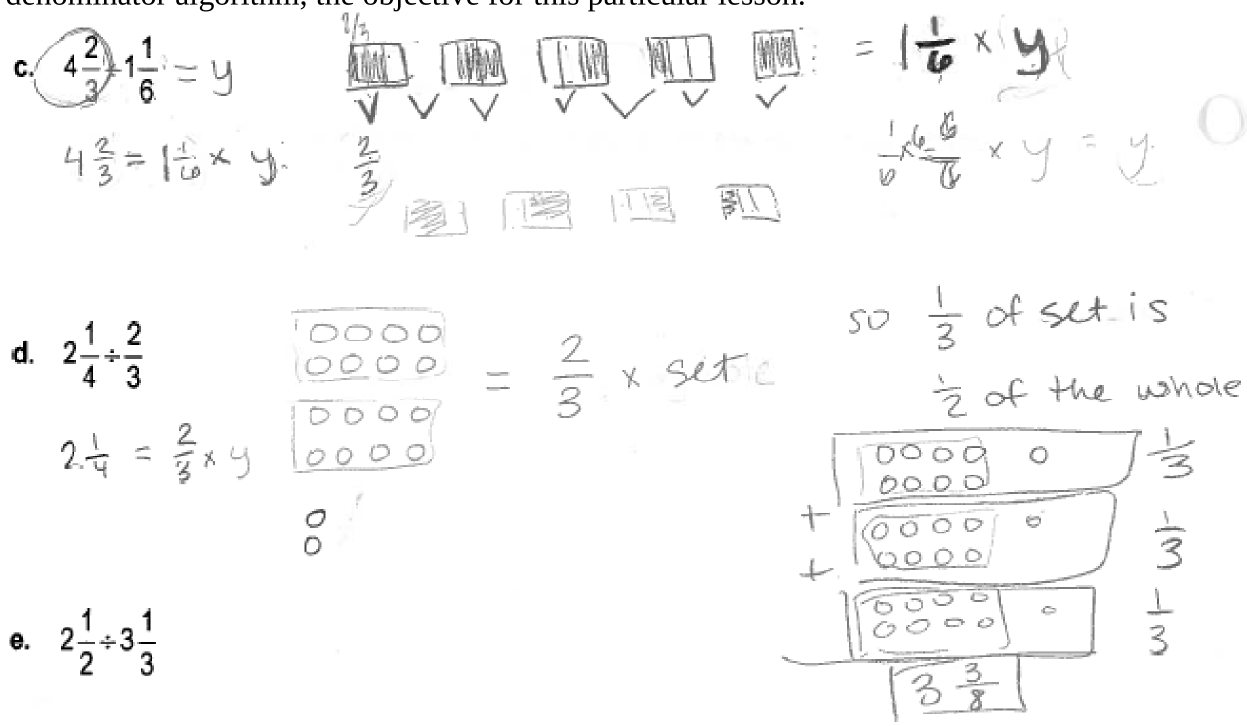

Figure 9.
Strategy using an equation for the third and fourth problems in the second class.

As expected, the students were confused; so, she also explained the use of her strategy for the fourth problem using colored chips (Figure 9). She again changed the problem to an equation, $2\frac{1}{4} = \frac{2}{3} \times y$, wanting to figure out the unit, $\frac{2}{3}$ of which is $2\frac{1}{4}$. Therefore, she determined $\frac{1}{3}$ (divided the chips into two equal groups) and then combined three of the $\frac{1}{3}$'s (multiply by three to get one whole). The other students asked why she used eight chips to represent a whole and she explained that she needed to be able to divide the chips that represented the $\frac{1}{4}$ into two groups. We acknowledged to the students that we, too, had difficulty understanding the approach and encouraged them to try to explain what she had done in their own words. If they got stuck at a certain point of their explanation, they should ask a question. A discussion generated by student-to-student questions ensued. The presenter also referred the class to questions they had answered previously in the course when we were building fraction sense (e.g. If 12 counters are three-fourths of a set, how many counters are in the full set? (Van de Walle & Lovin, 2006)). Although the questions were not identified as division problems, the same line of the thinking was used.

The students wanted to practice the strategy on an additional problem which I created on the spot ($3 \div \frac{2}{3}$). Since the class time was coming to an end, we attempted to connect the approach to the invert and multiply algorithm (divide 3, the dividend, by 2, the numerator, to get $\frac{1}{3}$ and multiply that value by 3, the denominator, to determine one whole). We could see that some of the students were still confused and said that we would explore the invert-and-multiply algorithm further in the next class.

To close, we requested that the other group that used the repeated addition strategy to explain how they discovered the common denominator algorithm. When solving the fourth problem, the students ran into trouble because the quotient was not a whole number. Initially, they expressed the quotient as $3\frac{3}{12}$ (Figure 10). They ultimately realized that the quotient was actually $3\frac{3}{8}$ by focusing solely on the numerators of the dividend and divisor when they were expressed with a common denominator. We ended by asking the other students in the class to outline the steps that are used in the common denominator algorithm.

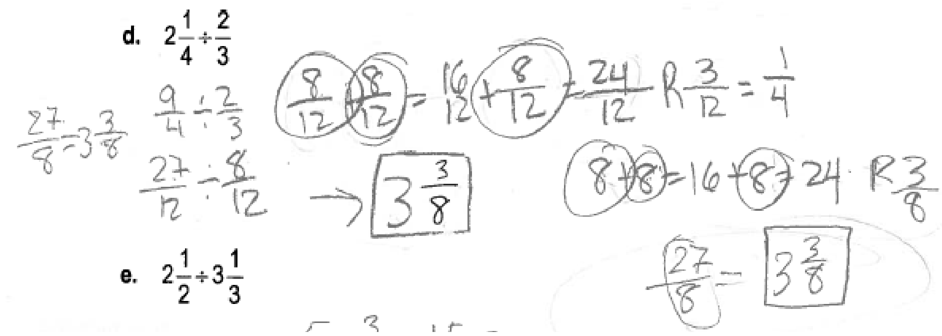

Figure 10.
Strategy using repeated addition for the fourth problem in the second class. (Note the incorrect use of the equal sign.)

Comparison of the Two Classes

Although we planned the same lesson for the two classes, the implementation of the lesson "looked" different. The whole-class discussions were constructed with the student-generated strategies, unique to each class. In the first class, the student work and subsequent conversation closely mirrored our predicted hypothetical learning trajectory. The approaches were similar to each other making it easier to draw upon their connections. Through the sequence of presentations, the students introduced the need to find a common denominator and the exchange culminated in the discovery of the common denominator algorithm. In comparison, the hypothetical learning trajectory in the second class was altered based on the student work. There was more diversity in the methods used by the students. Instead of realizing the common denominator algorithm through the whole-class discussion, the algorithm was presented as one of the strategies. Two of the groups had quickly figured it out by using repeated addition and then proceeded to solve the remainder of the problems using the newly discovered procedure. In addition, a student in one of the groups used a method which closely aligned with the invert-and-multiply algorithm. We considered whether to include this strategy because it was not our main learning objective for this particular class and it was challenging to understand. In the end, a significant portion of the discussion was centered on understanding this approach.

After reflecting on the classes, the question remained: Were the two implementations of the lesson, in fact, different? On the surface, the answer was yes; the students used different strategies which directly affected the whole-class discussion as described in detail. To truly address the inquiry, we needed to first remember our objectives for the course and the class. Overall, we wanted the pre-service teachers to develop a conceptual understanding of the mathematics that they will be teaching. we also wanted the students to understand and use multiple strategies to solve problems (invented and traditional algorithms) and to comprehend why those approaches work. To achieve these goals, we expected the students to communicate their ideas with their group members and the class as a whole as well as evaluate their work and that of others. For the class, we wanted to the students to realize the common denominator algorithm for dividing fractions. Although this discovery came about differently, the two classes were alike in many ways. In both situations, the students solved division-of-fraction problems in a nontraditional manner, explained their ideas to others, and examined and compared multiple strategies. The larger objectives for the course were met and, in this sense, the classes were similar.

Follow-Up Discussion with Pre-Service Teachers

Because this lesson was part of a course for pre-service teachers, we considered the challenges that we encountered in preparing for and conducting the class and thought about ways to support practitioners when teaching for understanding, an approach that might be outside of their comfort zone. As previously mentioned, through our instruction, we modeled methods that the students could use in an elementary classroom. The pre-service teachers experienced learning for understanding as students, but they must also understand this model of instruction from a teacher's perspective. Therefore, in the next class, we made pedagogical decisions explicit to the students.

To the pre-service teachers, it might appear as if we simply put five problems on a handout.We explained our process of preparing the lessons on division of fractions to dispel that myth. I sought out resources (e.g., Kribs-Zaleta, 2008; Van de Walle & Lovin, 2006; Warrington, 1997) about ideas on how to teach division of fractions and different ways that students thought about division-of-fraction problems, as Hiebert et al. (1997) suggested. Based on the literature, we selected five problems in a specific order to lead to our learning objective. In spite of our preparation, we was surprised by some of the strategies and acknowledged the vulnerability that we felt. The methods that we had not anticipated were now part of our, as well as the pre-service teachers', repertoire of how students think about division of fractions.

We also discussed with the pre-service teachers how we conducted the whole-class discussion. Hiebert et al. (1997) suggests that one of the roles of a teacher is to choose a sequence of tasks prior to a lesson. The role of the teacher changes during a lesson. In addition to observing and interacting with the students, the teacher also needs to make decisions about what students will present their strategies. "In doing so, the teacher reformulates and thereby implicitly legitimizes selected aspects of students mathematical contributions" (Cobb, Yackel, & Wood, 1992, p. 11). In comparison to choosing the tasks, this role can be more challenging because the teacher is making decisions in the moment and the student approaches might not be along the lines of the predicted hypothetical learning trajectory. As demonstrated in this article, each time a lesson is taught, the process of composing a whole-class discussion will change. As we chose the sequence of presentations, we considered the strategies to include, the order of the presentations, and my learning goals. we tried to incorporate all the groups' strategies into the conversation by building a progression of methods from the concrete to the abstract (Intel Math, 2008). In addition, we attempted to create a succession of approaches that culminated in the learning objective (e.g., the first class). If this was not possible, we modified the hypothetical learning trajectory (e.g., the second class). All of these decisions are based on the students' work and cannot be made prior to the lesson.

Preparing for and conducting this type of lesson is different than a traditional mathematics lesson. Instead of following a prescribed set of discrete lessons, the hypothetical learning trajectory, constantly readjusted, extends beyond a single class with the experiences in one lesson informing the next. The teacher needs to consider whether the culmination of experiences through a lesson, unit, and course build towards the learning objectives. Simon (1995) writes, "[A]s mathematics teachers, we strive to be purposeful in our planning and actions, yet flexible in our goals and expectations" (p. 141).

Conclusion

At face value, this article appears to be about division of fractions. However, this was only the context for considering the implications for a teacher when planning for and implementing a lesson in which the students learn for understanding. Pre-service teachers need to comprehend their new role from the perspective of a student and a teacher. They need to have a strong conceptual understanding of the mathematics so that they can identify the learning goals, access appropriate resources, choose activities, anticipate student thinking, comprehend student strategies, and compose whole-class discussions based on those approaches and the learning objectives. From year to year, a lesson implementation will be different as a teacher is working with new students. It is necessary for practitioners to become comfortable with situations that are, by nature, uncomfortable due the unpredictability of a teaching approach built upon students' ideas.

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