ADDING THE COMPLEMENT:

HOW AN ALTERNATIVE SUBTRACTION AGORITHM

LEAD TO A SUMMATION FORMULA FOR FINITE GEOMETRIC SEQUENCES

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Abstract

"Mathematics teachers are constantly being challenged to find problems that bring to their students the intellectual pleasure of searching, discovering, generalizing, and finding new and unexpected relationships" (Rulf, 1998, p.21). I have found this to be the case when teaching Math 201, Math for Elementary Teachers I. Math 201 is designed to familiarize the student with arithmetic concepts commonly taught in grades K-8. Students often perceive much of this course as a review, so I am always looking for problems that require conceptual understanding rather than an isolated knowledge of facts.

Introduction

While studying a section on whole number algorithms from their textbook, a class of aspiring elementary teachers was introduced to a subtraction algorithm called adding the complement. Unlike the standard algorithm for subtraction, this particular algorithm does not involve regrouping in the form of "borrowing." It simply requires that the student know addition facts for the numbers 0 through 9. During the process of studying this algorithm, my students posed and investigated questions eventually leading them to the summation formula for finite geometric series.

Algorithm: Adding the Complement

The subtraction algorithm, adding the complement, was introduced as a problem in a set of homework exercises in the required textbook for the course. Before examining examples of the algorithm, I reminded the class that in the subtraction problem $a - b$, a is the minuend, b is the subtrahend, and the number produced by $a - b$ is the difference. To produce a whole number difference, the minuend must be greater than or equal to the subtrahend. The algorithm requires that the following series of steps be completed.

1. Check to ensure that the subtraction problem produces a whole number difference. For example, 619 - 523 produces a whole number difference, since 619 > 523.

2. Find the number that when added to the subtrahend produces a sum in which each digit is 9. This number is called the complement and should contain the same number of digits as the minuend. In the problem 619 - 523, the complement is 476. Note that 523 + 476 = 999. Also, 999 is a three-digit number as is 619, the minuend.

3. Add the complement to the minuend. Using the example previously provided, 619 + 476 = 1095.

4. Cross out the leading digit in the sum. In the sum 619 + 476 = 1095, crossing out the leading digit in the sum is equivalent to eliminating the digit one appearing in the thousands place. This leaves the number 095 or 95.

5. Add one to the result from step 4. This final sum, 95 + 1 = 96, is equal to the original difference, 619 - 523.

Adding the complement also can be demonstrated using a more traditional vertical format. For example, to find 619 - 523:

619	095 (sum with leading digit crossed out
+476 (add the complement)	+1 (add one to the result)
1095	96

Thus, the original difference is 96.

Searching and Discovering

After my demonstration of how to use the algorithm, many students became excited and immediately started asking questions about it. The first question posed by several students in the class was, "Will this method work for other types of subtraction problems?" In other words, do the number of digits in both minuend and subtrahend have to be the same?

In an attempt to answer this question, the class decided to examine several specific cases and search for a pattern. Since this course is typically taught using a lecture-discussion format to introduce topics and small group activities for the reinforcement of concepts and skills, my students were comfortable participating in both large and small group discussions. To investigate specific cases, the class decided to break into smaller groups consisting of four to six students per group. One group of students elected to study one-digit minus one-digit, another group studied two-digit minus two-digit, and another, four-digit minus four-digit subtraction problems. The remaining group decided to examine four-digit minus two-digit problems. The following results were obtained:

one-digit – one-digit problems

9 - 5 = [(9 + 4) - 10] + 1 = 4
7 - 3 = [(7 + 6) - 10] + 1 = 4

two-digit – two-digit problems
64 - 39 = [(64 + 60) - 100] + 1 = 25
88 - 57 = [(88 + 42) - 100] + 1 = 31

four-digit – four-digit problems
3721 - 2876 = [(3721 + 7123) - 10,000] + 1 = 1845
6972 - 4395 = [(6972 + 5604) - 10,000] + 1 = 2577

four-digit – two-digit problems
3721 - 56 = [(3721 + 9943) - 10,000] + 1 = 3665
8952 - 95 = [(8952 + 9904) - 10,000] + 1 = 8857

Each small group presented their findings to the class as a whole. After examining the specific cases chosen, students made the following observations.

1. The algorithm works without regard to the number of digits in either minuend or subtrahend.
2. When the highest power in the minuend is 10^n, crossing out the leading digit in the sum is the same as subtracting 10^{n+1}.

This discussion served to stimulate the interest of those students who were previously only interested in how to use the algorithm to find an answer to a subtraction problem. It also prompted further investigation into how the algorithm works. More specifically, students wanted to be able to see exactly why they had to cross out 10^{n+1} and add one. To answer this question, the students applied the commutative and associative properties of whole number addition and the distributive property of multiplication over addition to the original problem 619 - 523 as follows:

619 - 523 = 619 - (999 - 476) by renaming 523
= 619 - [(1000 - 1) - 476] by renaming 999
= 619 - (1000 - 1 - 476)
= 619 - 1000 + 1 + 476 by using the Distributive Property
= 619 + 476 - 1000 + 1 by using the Commutative Property
= [(619 + 476) - 1000] + 1 by using the Associative Property

Whole number properties had been introduced in an earlier chapter in the textbook and were memorized, under protest, by students with little regard for when, if ever, they would need to know these properties. Some of the students commented that they felt a sense of satisfaction in being able to use and identify properties in the process of answering a question that they had posed. Thus, this exercise served to validate the usefulness of being familiar with whole number properties.

Generalizing

One student from the group requested a more in-depth look at the algorithm. This particular student had expressed his interest in math and had shown an aptitude for mathematics earlier in the semester. He wanted to know if it was possible to write out the algorithm using variables and to show, in an abstract fashion, that the algorithm would always work. His request resulted in anxiety, masked as apathy, on the part of many of his classmates. The prerequisite for Math 201 is high school Algebra II or its equivalent. Since the majority of students had a somewhat limited algebra background, and because I had never looked at the algorithm in this way before, this particular problem became an optional, out of class, challenge project.

A couple of students struggled for several days with the problem. At the same time, I found myself toying with a few ideas of my own. All of the students who worked on this challenge problem found it difficult to represent the minuend, subtrahend, and complement abstractly using place value. After a two-week investigation period, the following was presented to the class by me.

Minuend = $a_n(10^n) + a_{n-1}(10^{n-1}) +...+ a_2(10^2) + a_1(10) + a_0(1)$

Subtrahend = $b_n(10^n) + b_{n-1}(10^{n-1}) +...+ b_2(10^2) + b_1(10) + b_0(1)$

Complement = $(9 - b_n)10^n + (9 - b_{n-1})10^{n-1} +...+ (9 - b_2)10^2 + (9 - b_1)10 + (9 - b_0)1$

Minuend + Complement

$= (a_n - b_n + 9)10^n + (a_{n-1} - b_{n-1} + 9)10^{n-1} +...+ (a_2 - b_2 + 9)10^2 + (a_1 - b_1 + 9)10 + (a_0 - b_0 + 9)1$

$= (a_n - b_n)10^n + (a_{n-1} - b_{n-1})10^{n-1} +...+ (a_2 - b_2)10^2 + (a_1 - b_1)10 + (a_0 - b_0)1 + 9(10^n) + 9(10^{n-1}) +...+ 9(10^2) + 9(10) + 9(1)$,

by using commutativity and associativity.

Minuend + Complement $- 10^{n+1} + 1$

$= (a_n - b_n)10^n + (a_{n-1} - b_{n-1})10^{n-1} +...+ (a_2 - b_2)10^2 + (a_1 - b_1)10 + (a_0 - b_0)1 + 9(10^n) + 9(10^{n-1}) +...+ 9(10^2) + 9(10) + 9(1) - 10^{n+1} + 1$

$= (a_n - b_n)10^n + (a_{n-1} - b_{n-1})10^{n-1} +...+ (a_2 - b_2)10^2 + (a_1 - b_1)10 + (a_0 - b_0)1 + 9(10^n) + 9(10^{n-1}) +...+ 9(10^2) + 9(10) + (9+1) - 10^{n+1}$,

using commutativity and associativity so that we can add 9+1 to obtain 10.

Since we added 9+1 to make 10, this gave us 10(10), which resulted in $10(10^2)$, and so on, until 10^{n+1} was produced as a final result. Thus,

Minuend + Complement $- 10^{n+1} + 1$

$= (a_n - b_n)10^n + (a_{n-1} - b_{n-1})10^{n-1}+...+ (a_2 - b_2)10^2 + (a_1 - b_1)10 + (a_0 - b_0)1 + 10^{n+1} - 10^{n+1}$

$= (a_n - b_n)10^n + (a_{n-1} - b_{n-1})10^{n-1}+...+ (a_2 - b_2)10^2 + (a_1 - b_1)10 + (a_0 - b_0)1$

= Minuend - Subtrahend.

New and Unexpected Relationships

The final question asked by one student was whether or not this was the only way to write out the algorithm. I took this opportunity to point out that the sum

$9(10^n) + 9(10^{n-1}) + ... + 9(10^2) + 9(10) + 9(1)$ is a finite geometric series with a common ratio, r, of 10.

The class had previously been introduced to the notion of finite sequences during a brief study of the problem solving strategy of looking for a pattern. The students were not familiar with geometric sequences or sums, nor had they had any exposure to nth term or summation formulas.

To find the sum, $9(10^n) + 9(10^{n-1}) + ... + 9(10^2) + 9(10) + 9(1)$, we started by calling this sum S_n. We then multiplied the sum by 10, giving

$10S_n = 9(10^{n+1}) + 9(10^n) + ... + 9(10^3) + 9(10^2) + 9(10)$.

$10S_n - S_n = 9S_n = 9(10^{n+1}) - 9(1) = 9(10^{n+1} - 1)$. So, $S_n = 10^{n+1} - 1$.

This result for S_n was used to rewrite Minuend + Complement - $10^{n+1} + 1$ as

$= (a_n - b_n)10^n + (a_{n-1} - b_{n-1})10^{n-1} + ... + (a_2 - b_2)10^2 + (a_1 - b_1)10 + (a_0 - b_0)1 + S_n - 10^{n+1} + 1$

$= (a_n - b_n)10^n + (a_{n-1} - b_{n-1})10^{n-1} + ... + (a_2 - b_2)10^2 + (a_1 - b_1)10 + (a_0 - b_0)1 + 10^{n+1} - 1 - 10^{n+1} + 1$

$= (a_n - b_n)10^n + (a_{n-1} - b_{n-1})10^{n-1} + ... + (a_2 - b_2)10^2 + (a_1 - b_1)10 + (a_0 - b_0)1$

= Minuend - Subtrahend.

By this time, my students had become very comfortable with the notation being used for the minuend, subtrahend, and complement. Since they no longer seemed intimidated by the abstract representation of these numbers, I decided it would be a worthwhile project to derive a general summation formula for geometric series.

I began by giving the class the following definition of a geometric sequence: A sequence is geometric if each pair of successive terms has the same ratio, r, such that $r = a_{i+1}/a_i$, for r not equal to 0. The number r is called the common ratio.

We noted that the terms of a geometric sequence would be a, ar, ar^2, ar^3, ... ar^{n-1}, ar^n. The class also determined that the sum we were interested in would look like $S_n = ar^n + ar^{n-1} + ... + ar^2 + ar + a$.

Since the method used in the previous sum was to multiply the sum by the common ratio, r = 10, we multiplied S_n by r. Now,

$rS_n = ar^{n+1} + ar^n + ... + ar^3 + ar^2 + ar$. So,

$rS_n - S_n = S_n(r - 1) = ar^{n+1} - a = a(r^{n+1} - 1)$. Therefore,

$S_n = [a(r^{n+1} - 1)]/(r - 1)$ for r not equal to 1.

Summary

Following a demonstration of how to use the subtraction algorithm of adding the complement, my Math 201 students posed and investigated several questions regarding why this algorithm works. To answer their questions, students had to “organize their knowledge in a simple, coherent, and logical manner." In other words, my students had to "think mathematically" (Solow, 1995, p.1). As a result of this mathematical adventure, many students who had previously expressed disinterest in and a dislike for mathematics, professed a new-found appreciation for the process of mathematical discovery. Since teacher attitude is an important factor in student learning, this is something that, hopefully, will carry over into their own teaching.

References

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