Developing Prospective Teachers' Understanding of Addition and Subtraction with Whole Numbers

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Abstract

*This study was situated in a semester-long classroom teaching experiment examining prospective teachers' understanding of number concepts and operations. The purpose of this paper is to describe the learning goals, tasks, and tools used to cultivate prospective teachers' understanding of addition and subtraction with whole numbers. Research regarding children's understanding of whole number concepts and operations was used in developing learning goals, pathways for learning, and instructional tasks for the prospective teachers. Furthermore, prior research regarding prospective teachers' whole number development supported the expectation that all of the instructional tasks were reasoned solely in base-eight. Classroom episodes indicate that base-eight allows prospective teachers to reason about addition and subtraction with whole numbers in similar ways that elementary aged students' reason in base-*ten.

Keywords: Prospective Teachers, Addition, Subtraction, Whole Numbers

Introduction

Whole number concepts and operations are part of the core mathematics content of elementary schools in the United States (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Yet research has shown that some elementary teachers lack conceptual understanding of these fundamental concepts (Ma, 1999). Beliefs held by many prospective elementary teachers include the notion that learning mathematics is no more than following a set of rules or predetermined steps, a view that is in direct conflict with a more conceptual approach to mathematics teaching and learning (Philipp, et al., 2007). Additionally, some prospective teachers (PSTs) believe that if they do not already know an elementary mathematical concept and they are in college, then the children they instruct will not be expected to know it (Phillip et al., 2007). These beliefs neglect the specialized knowledge necessary to successfully teach mathematics (Hill, Schilling, & Ball, 2004; Ma, 2009). Similar to previous research (Andreasen, 2006; McClain, 2003; Yackel, Underwood, & Elias, 2007), this study was guided by research regarding elementary aged children's understanding of addition and subtraction with whole numbers.

Children's Understandings of Addition and Subtraction with Whole Numbers

Research has shown that elementary aged children develop a variety of strategies when reasoning with whole number addition and subtraction context problems (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Carpenter, Hiebert, & Moser, 1983). Initially, when given context problems, many children attempt to directly model the action or relationship with physical objects or drawings that exists in a problem. As their thinking matures, children eventually transition from direct modeling strategies, using physical objects, to counting strategies without direct modeling (Carpenter & Moser, 1984). These counting strategies include counting-on from the first addend and counting-on from the larger addend. Finally, children begin to represent solutions to problems that are not consistent with the structure of the problem (Carroll & Porter, 1998). At this point, children develop flexible procedures relying more on recalled and derived number facts and less on the established computational algorithms taught in most elementary classes (Carpenter & Moser, 1984).

When children are encouraged to develop their own meaningful computational strategies prior to algorithms being introduced, they demonstrate their mathematical understanding rooted in number sense and place value (Beishuizen, 1993; Kamii & Joseph, 1988; Madell, 1985; Selter, 2002; Thompson, 1994). After interviewing elementary students, Carroll and Porter (1997) concluded that the students' idiosyncratic tendencies often do not fit the standard algorithms taught in U.S. schools. Carroll and Porter (1998) also noted that it is frequently beneficial to allow students to derive their own algorithms because different problems are better suited to work with certain numbers. This point of view is supported by Thompson (1994) who reported that elementary aged students preferred adding from left to right. When doing so, the children developed both a *partial sums* strategy where students would add according to the addends place value and a *cumulative sums* strategy where the students would calculate a running sum when adding portions of a decomposed second addend. Similarly, Beishuizen (1993) identified analogous left to right mental strategies for addition. Among the strategies were a *split* strategy where addends are added according to place values, and a *jumping* strategy where a collective sum is amassed. Finally, Selter (2002) described three-digit *number* strategies. Among these number strategies rooted in place value was a *hundreds, tens, units* strategy where children would perform the operation in pieces according the place value; and the *stepwise* strategy in which a child would add or subtract the second number portion by portion beginning with the hundreds portion of the number and ending with the ones portion of the number.

Moreover, these findings document that elementary aged students are able to achieve computational flexibility by using a variety of procedures or strategies. As Ma (1999) emphasized, "being able to calculate in multiple ways means that one has transcended the formality of the algorithm and reached the essence of the numerical operations—the underlying mathematical ideas and principles" (p. 112). Since this flexibility demands more of children than simply following steps to compute, methods and procedures should be tools to solve problems rather than the goals of mathematics instruction (National Research Council, 2001). Consequently, when children are allowed to invent addition and subtraction strategies, they strengthen their mathematical connections between place value, estimation, number sense, and properties of operations (Beishuizen, 1993; Carroll & Porter, 1998; Huinker, Freckman, & Steinmeyer, 2003; Kamii, Livingston, & Lewis, 1993; Madell, 1985; Selter, 2002; Thompson, 1994). Given that children's strategies for whole number addition and subtraction can be complex and varied, PSTs must possess mathematical understandings equally complex and varied.

Prospective Teachers' Explorations of Addition and Subtraction with Whole Numbers

In order to teach children in the future, a PST should possess more than the ability to perform an algorithm (Thanheiser, 2009), especially since some believe learning mathematics is no more than following a set of rules in a predetermined systematic fashion (Philipp et al., 2007). However, their familiarity with base-ten can prevent them from deeply exploring some whole number concepts they should have learned as children (Hopkins & Cady, 2007). One method used by mathematics educators to circumvent this familiarity and to create cognitive dissonance is to situate whole number tasks in numeration systems other than base-ten; these number systems include base-five, base-eight and base-twenty (Andreasen, 2006; Cady, Hopkins, & Hodges, 2008; McClain, 2003; Thanheiser & Rhoads, 2009; Yackel, Underwood, & Elias, 2007). Although the PSTs are not experiencing whole number concepts and operations for the first time as children would be, by exploring whole number concepts in this manner, prospective teachers are expected to explore developmental pathways that are similar to ones elementary children experience.

As in previous research (Andreasen, 2006; McClain, 2003; Yackel, Underwood, & Elias, 2007) base-eight was leveraged in this study primarily because it was unfamiliar to prospective teachers while still mimicking number patterns that occur in base-ten. However, results from previous studies cautioned that a potential reliance on base-ten could surface (Andreasen, 2006; Cady, Hopkins, & Hodges, 2008; McClain, 2003). As PSTs teachers solved problems in these studies, they developed tricks using base-ten to solve problems posed in other bases. As a result, they manipulated symbols rather than attempting to understand the numerical quantities and mathematical connections when solving problems (Cady, Hopkins, & Hodges, 2008; McClain, 2003). To avoid this type of symbolic manipulation, all of the whole number instructional tasks described in this paper were posed in base-eight with the explicit expectation that the PSTs solve the problems solely using base-eight, and not convert from base-eight to base-ten and then back to base-eight.

Method

The whole number learning goals in this part of study were based upon the big ideas through which elementary aged children progress when developing proficiency with respect to whole number concepts and operations. The tasks in the instructional sequence were grouped into three phases around the mathematical concepts shown in Table 1.

Table 1
Initial Instructional Sequence for Whole Number Concepts

Phase	**Learning goal**
One	Count and unitize objects efficiently
Two	Flexible representations of numbers
Three	Computational Strategies

At each phase of the instructional sequence, the course instructor engaged PSTs in instructional tasks needed to develop the prospective teachers' understanding of whole numbers. To aid in this mathematical development, the course instructor introduced various *pedagogical content tools* (Rasmussen & Marrongelle, 2006). A pedagogical content tool is a notation, diagram, or graphical representation of one's thinking that can be used to answer new problems. These tools allowed PSTs to describe their mathematical thinking and simultaneously allowed the instructor to emphasize learning goals. The focus of this paper is to describe the learning goals, tasks, and tools used to cultivate prospective PSTs' understanding of addition and subtraction with whole numbers.

Setting and Participants

This study was situated within a classroom teaching experiment in an elementary education mathematics content course taught in a college of education at a major university located in the southeastern United States. Participants in the study included 33 female PSTs in their junior or senior academic year majoring in elementary education or exceptional education. Class sessions were held twice per week; each session was one hour and fifty minutes long. Ten class sessions were devoted to the whole number concepts and operations instructional unit (for details regarding complete whole number instructional sequence see

Roy, 2008); of these ten class sessions, six were devoted to counting, addition, and subtraction whereas the other four sessions were devoted to multiplication and division.

During each class session of the whole number concepts and operations instructional unit, the PSTs were posed with mathematical tasks in three phases. First, the instructor launched a task by presenting a base-eight mathematical scenario in the form of a word problem, picture, or both. Next, the prospective teachers solved the problems in ways that made sense to them mathematically. The prospective teachers worked either individually or in groups of two to four individuals. Finally, the course instructor facilitated whole-class discussions in order to allow the PSTs an opportunity to examine different solutions and strategies.

Data Collection/Analysis

Each of the ten class sessions were videotaped to record the whole class dialogue that occurred between the instructor and the PSTs, and between the PSTs themselves. These video records were then transcribed for analysis. The transcripts were then independently coded by at least two members of the research team to identify and verify classroom episodes according to the mathematical tasks. The episodes were identified by the instructional task with which the PSTs were engaged. These episodes were then compared to previous and subsequent episodes to ascertain normative lines of reasoning. The findings illustrate a synthesized overview documenting the PSTs' collective thinking about whole number addition and subtraction. This methodology is consistent with the four-stage constant comparative method described by Glasser and Strauss (1967) and the methodology used for conducting longitudinal analysis described by Cobb and Whitenack (1996).

Findings

Since the PSTs were unfamiliar with base-eight, initial tasks were created to support familiarity of base-eight through counting. To accomplish this, context problems were assigned with the expectation that the PSTs' solutions were completely reasoned using base-eight.

In order to aid prospective teachers in maintaining this expectation, names of base-eight numbers were developed to differentiate between the base-eight and base-ten number systems. When introducing base-eight nomenclature, the instructor had the prospective teachers count and skip count. When doing so the PSTs were confronted with 10_8 and 100_8. For example, when counting by 1_8 beginning with 76_8 and counting to 102_8, the prospective teachers would say sevenee-six (76_8), sevenee-seven (77_8), one-hundree (100_8), one hundree-one (101_8), one-hundree-two (102_8). When reading this paper all base-eight numbers should be understood to follow this terminology.

Furthermore, the instructor and prospective teachers negotiated social norms such as explaining and justifying a solution, and making sense of a peer's solution, and the sociomathematical norms: acceptable and different solutions (Cobb & Yackel, 1996; Roy, Tobias, Safi, & Dixon, 2010). In the end these norms were vital in creating the mathematical climate needed to address the learning goals of the instructional sequence.

Counting Strategies in Context

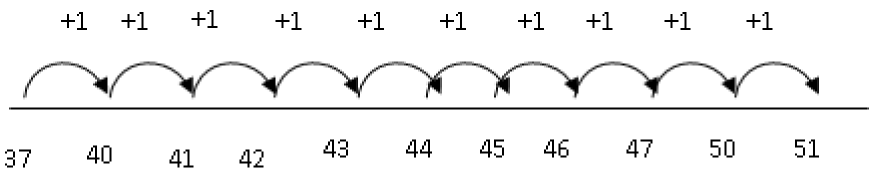
To assist the PSTs in describing their counting strategies in base-eight, the instructor introduced an *empty number line* (Gravemeijer, 1994; Klein, Beishuizen, & Treffers, 1998; Treffers, 1991). The empty number line begins without any numbers recorded on it; as an individual records their thinking, they fill in numbers on the number line. Two unique ways of counting surfaced when using the empty number line: (a) counting by 1_8; and (b) counting by groups of 10_8 and 1_8. The dialogue that follows documents each of the ways PSTs reasoned using the empty number line.

Counting by 1_8. In the following episode, Cordelia articulated the method in which she reasoned to solve the following join-result unknown context problem type (Carpenter, Fennema, et al., 1999).

Marc had 12_8 roses. He bought 37_8 more. How many did he buy altogether?

As Cordelia described her reasoning, the instructor simultaneously recorded her thinking on the whiteboard completing the empty number line shown in Figure 1.

Figure 1:
Cordelia's representation of solving $12_8 + 37_8$.

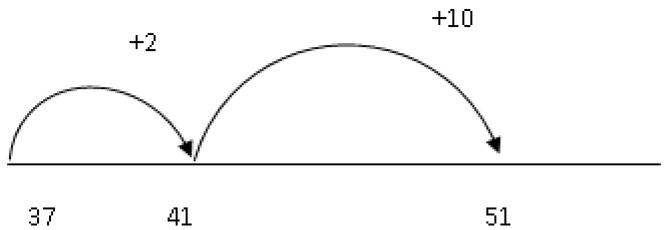


Cordelia: I counted from the 37_8, 40_8, 41_8, 42_8…
Instructor: So you counted by ones on this.
Cordelia: … 45_8, 46_8, 47_8 …

When using a counting by 1_8 strategy to solve the context probem, Cordelia begins with the greater of the two addends and counts on each individual rose until she arrives at her result, 51_8. This counting on from the greater addend strategy allowed Cordelia to easily accumulate the start value in the problem, 12_8, from the change, 37_8, value while crossing the decade numbers 40_8 and 50_8. PSTs identified and described solution processes including the counting strategies *counting up by 1_8*, and *counting on from the larger* described by Baroody (1987) in his study with kindergarten children.

Counting by Groups of 10_8 and 1_8. Other PSTs utilized more efficient counting strategies when solving the problem. For example, Claire represented her thinking by *counting by 10_8 and 1_8* on the empty number line as shown in Figure 2.

Figure 2:
Claire's representation of $12_8 + 37_8$



Claire: 37_8 plus 2_8.
Instructor: Okay, so if I do 37_8 plus 2_8.
Claire: It is 41_8, then you add 10_8, and it would be 51_8.

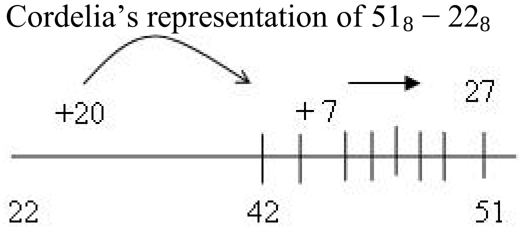
In her explanation, Claire found it easier to decompose 12_8 by place value into 2_8 and 10_8, and then add each of the decomposed parts to 37_8 to find the unknown result. In Claire's strategy, she did not find it necessary to add units individually as Cordelia did in the previously presented strategy. As such, Claire provided the class with a different and more sophisticated *jumping* or *cumulative sums* or *stepwise* strategy that children use to solve problems in base-ten (Beishuizen, 1993; Bobis, 2007; Selter, 2002; Thompson, 1994).

As the PSTs became more comfortable solving base-eight context problems, their solution strategies using groups of 10_8 and 1_8 became more inventive. In the following episode, the PSTs discussed solution strategies to the following separate context problem where the result is unknown (Carpenter, Fennema, et al., 1999).

> ***There were 51_8 seagulls on the beach, and 22_8 flew away. How many are still on the beach?***

As with previously presented counting strategies, a common practice for members of the class was to use the empty number line to represent and support their reasoning, see Figure 3.

Figure 3:
Cordelia's representation of $51_8 - 22_8$

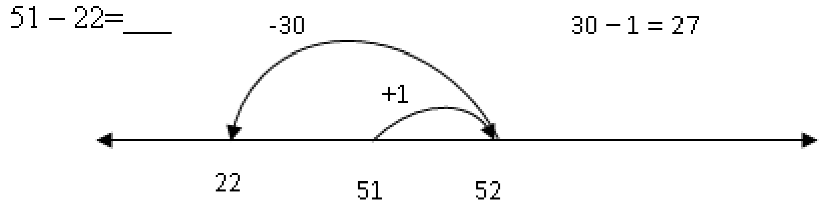


Cordelia: Well I started with just 22_8 and I went plus 20_8 to get 42_8 and then I went 43_8, 44_8, 45_8, 46_8, 47_8, 50_8, 51_8 and I counted my lines, 1_8, 2_8, 3_8, 4_8, 5_8, 6_8, 7_8, and I added 7_8 to my 20_8, 27_8.

As Cordelia communicated her thought process, her own reasoning in base-eight evolved and was quite different from the "*counting by 1_8*" strategy she presented when solving the previous problem [Figure 1]. Initially, the dissimilarity was subtle as Cordelia also drew a number line that accounted for two groups of 10_8 and seven 1_8s. However, instead of placing 51_8 on the number line and finding the difference between the numbers by using subtraction, she instead counted up from 22_8 to 51_8. Cordelia's counting on strategy exhibited mathematical connections between the inverse operations of addition and subtraction, this type of counting on strategy to find the difference became one that many of the PSTs employed when solving subtraction problems.

Another strategy emerged as Edith described her reasoning about the same problem, shown in Figure 4.

Figure 4:
Edith's representation of $51_8 - 22_8$



Edith: I saw that it was 51_8 seagulls minus 22_8. And I realized that, if I added one to 51_8 I could easily from that number subtract 30_8 to get 22_8. So then all I did was take 30_8 minus the 1_8 that I added originally and got 27_8.

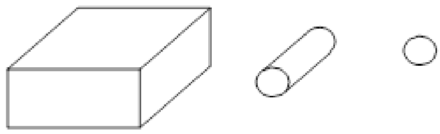
To start the problem, Edith placed both 22_8 and 51_8 on the number line realizing that the difference was the amount between both numbers. Edith then used her understanding of inverse properties and part whole relationships to add 1_8 to the number of seagulls on the beach to solve the problem “more easily.” Edith finalized her thinking by subtracting 1_8 at the end of the problem in order to obtain the distance between 51_8 and 22_8. This type of compensation subtraction strategy (Bobis, 2007) became one used by several other members of the class.

In the presented counting strategies and their empty number line representations, prospective teachers used mathematical connections between composition of numbers and properties of numbers to solve various addition and subtraction context problems; this is in contrast to previously conducted research, where some prospective teachers missed these connections by only focusing on the ability to perform an algorithm or used tricks to solve computational problems (McClain, 2003).

Strategies Emphasizing Unitizing

Although it appeared PSTs made place value connections in their counting strategies using the empty number line, it was unclear to the research team if they were unitizing quantities when using 10_8 in their reasoning. More specifically, from their reasoning there was no evidence that the PSTs simultaneously viewed 10_8 as one group and as 10_8 individual units (Cobb & Wheatley, 1988; National Council of Teachers of Mathematics, 2000) when using the empty number line. As a result, during the second phase of the instructional sequence, the instructor introduced instructional tasks emphasizing PSTs’ ability to flexibly represent a number. The instructor did so by having them investigate equivalence through transformation tasks (Yackel, Underwood, & Elias, 2007) that supported regrouping. In order to support this mathematical development, the instructor introduced a *Candy Shop Scenario* previously used with children (Cobb, Boufi, McClain, & Whitenack, 1997; Cobb, Yackel, & Wood, 1992; Wood, 1999). Prospective teachers were asked to package candy in Boxes, Rolls, and Pieces as shown in Figure 5; 10_8 pieces = 1_8 roll and 10_8 rolls = 1_8 box.

Figure 5:
Candy Shop Package Types

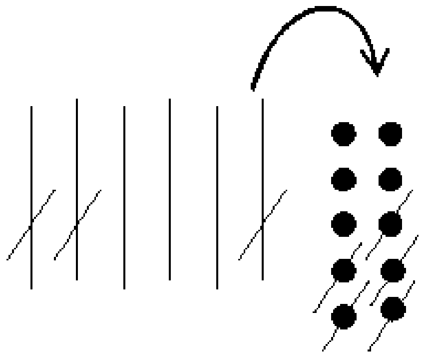


The PSTs in the class used their understanding of unitizing and the 10_8 to 1_8 relationship to compose or decompose equivalent quantities. This is important because the mathematical emphasis was on not only the package type, but also the quantity that the package contained. Three strategies using boxes, rolls, and pieces emerged as students solved the following separate-change unknown context problem (Carpenter, Fennema, et al., 1999).

> ***There were 62_8 lemon candies in the candy shop. After a customer bought some there were only 25_8 lemon candies in the shop. How many lemon candies did the customer buy?***

In the following solution presented, Claire used Candy Shop terminology as she described using rolls and pieces to solve the presented transaction task (Yackel, Underwood, & Elias, 2007). The instructor recorded her thinking and drew the picture represented in Figure 6.

Figure 6:
Claire's solution

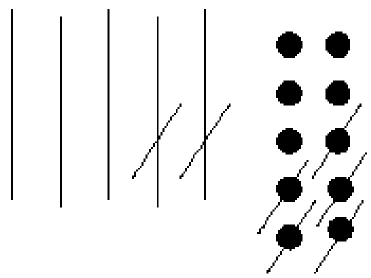


Claire: I drew 6_8 rolls and 2_8 pieces. And then to find out how many there were I took away 25_8, so I had to convert 1_8 of the rolls to pieces.
Instructor: So you didn't draw 25_8 here, you just took it away from here?
Claire: I made that 1_8 roll into pieces, and then I took away 2_8 rolls.
Instructor: So you took the rolls away first.
Claire: Yeah, and then I crossed out 5_8 pieces. And that left me with 35_8.

In her explanation, Claire implemented a strategy that was based on finding equivalent amounts of candy. Because Claire was unable to remove 25_8 candies from 62_8 without regrouping, she converted 1_8 roll into pieces. Claire then performed the operation by addressing the larger portion of the result when she subtracted the 2_8 rolls followed by 5_8 pieces from the decomposed total 5_8 rolls and 12_8 pieces to arrive at her solution, 35_8.

Edith also described using rolls and pieces to solve the context problem; however, instead of decomposing a roll into pieces, Edith composed 62_8 as 5_8 rolls and 12_8 pieces. The picture Edith used is represented in Figure 7.

Figure 7:
Edith's solution



Edith: I started out, I knew that if I drew, 6 rolls I would have to break them down, like break one apart to get the 5_8 pieces for 25_8, so I just drew 5_8 rolls and 12_8 pieces, and then I took away the 5_8 pieces and then took the 2_8 rolls.
Instructor: So what was different about how Edith solved the problem? Jane.
Jane: She did pieces first.
Instructor: She did, she took away her pieces first. What else is different? Jackie.
Jackie: She didn't draw the 6_8 rolls and then convert 1_8 into pieces. She just kind of did that in her mind, she drew 5_8 rolls to start with.

The roots of each of these flexible transaction strategies can be traced back to counting by 10_8 and 1_8 using the empty number line, and flexibly representing equivalent quantities using boxes, rolls and pieces found in the Candy Shop.

Conceptual Addition and Subtraction Algorithms

The final learning goal of the instructional sequence was to assist the prospective teachers in developing procedural fluency or efficient, flexible, and accurate strategies for computing (National Council of Teachers of Mathematics, 2000, National Research Council, 2001). The instructor introduced an Inventory Form, shown in Figure 8 as a more efficient way to record and support learning. This tool was used to document transactions as candy totals change (Yackel, Underwood, & Elias, 2007) as in the following join-result unknown context problem (Carpenter, Fennema, et al., 1999).

Figure 8:
Join-result unknown context problem

This many candies were in the store room

Boxes	Rolls	Pieces
2	3	6

The factory makes 146_8 more candies. How many candies are in the store room now?

The introduction of the Inventory Form led PSTs to an algorithmic approach to solving the context problems; for example, Claire demonstrated her reasoning in the representation shown in Figure 9.

Figure 9:
Claire's addition algorithm

B	R	P
2 + 1 = 3	3 + 4 = 7	6 + 6=14
3	7	~~14~~
3	~~10~~	4
4	0	

Instructor: And then you said you converted them; to keep track of them? Okay. … Is that the order you did it?

Claire: Well I guess, okay, no, okay, and then I turned into numbers and then I worked with the numbers, so I had the 14_8 pieces, the 7_8 rolls and the 3_8 boxes, and then I did what we did with the Inventory Forms, and I put the pieces to rolls, and rolls to boxes.

In her solution, Claire provided a mathematical justification, "I put the pieces to rolls, and rolls to boxes," connecting her method of solving the problem, and as a result described the computation conceptually. Although, she did not realize it, Claire invented her own column procedure to add the numbers (Kammii & Joseph, 1988; Madell, 1985). Claire's process was followed by Caroline's solution process using Inventory Forms shown in Figure 10.

Figure 10:
Caroline's addition algorithm

Boxes	Rolls	Pieces
1	1	
2	3	6
+1	4	6
4	0	4

Caroline: I wrote 2_8 for the boxes, 3_8 for the rolls, 6_8 for the pieces. 1_8 for the boxes, 4_8 for the rolls, 6_8 for the pieces, and I added starting from the pieces.
Instructor: What did you write down here?
Caroline: I wrote 4_8 pieces, and I added a roll.
Instructor: So you never wrote the 1_8 here [referring to the sum in the pieces]?
Caroline: No; I just added 1_8 onto the rolls.
Instructor: Okay.
Caroline: And then I got 10_8 rolls, so I just wrote 0, and put 1_8 for the boxes.
…
Caroline: So yeah. And it's okay to carry those over because 10_8 pieces equals 1_8 roll, and so instead of writing the 14_8, I wrote 1_8 roll and 4_8 pieces. The same thing with the rolls to boxes, I converted them.

In each of the presented solution methods, the students described conceptual ways of solving the problem using place value. Claire described how she added in columns prior to regrouping her boxes, rolls, or pieces, while Caroline described a traditional addition algorithm method utilized in the United States using conceptually appropriate language emphasizing unitizing.

These conceptual justifications were exhibited even when explaining non-context problems, like the one shown in Figure 11.

Figure 11:
$417_8 - 253_8$ in an Inventory Form

Boxes	Rolls	Pieces
4 -2	1 5	7 3

Olympia: Okay. Oh, since we started out with 7_8; it was a little easier I guess. I know 3_8 from 7_8 is 4_8, and then …
Instructor: Put the 4_8 here [indicated the pieces column]?
Olympia: Yes. I know I can't take away 5_8 from 1_8 so I have to break up a box. And when you break down the box, you take away 1_8 box, cause you're breaking down 1_8 box, so now you are left with 3_8 in the box column, and it becomes 10_8 rolls in the rolls column, no it doesn't, you, you bring 10_8 rolls over so now it becomes 11_8 rolls. And now, you take the 5_8 rolls from the 11_8 rolls, that becomes 4_8, and 2_8 boxes from 3_8 boxes is 1_8. And now I checked it with addition, and I did 2_8 boxes, 5_8 rolls, 3_8 pieces plus 1_8 box, 4_8 rolls, 4_8 pieces. I know 3_8 plus 4_8 is 7_8, and 5_8 plus 4_8 is 11_8, and I know 2_8 boxes plus 1_8 box is 3_8, and … I added the 1_8 in the, the first one, with the 3_8, and I got 4_8 so it becomes 4_8 boxes, 1_8 roll, 7_8 pieces.

In the presented dialogue, Olympia determined the accuracy of her answer to a subtraction problem with addition. In her "check," Olympia described the addition steps she used to arrive at the minuend, 417_8. As PSTs' reasoned with Inventory Forms, their addition or subtraction strategies folded back to counting strategies such as counting on by groups using an empty number line and flexibly representing numbers using boxes, rolls, and pieces.

Discussion

The development of PSTs' understanding of whole number concepts and operations was emphasized in this study for two reasons. First, since it is a core component of elementary school mathematics, its importance is inherent to future mathematics learning (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Second, PSTs are entering the profession lacking the depth of knowledge necessary to support the unique understandings children have of whole number concepts and operations (Ball, 1990).

The addition and subtraction strategies presented in this paper document the maturation and flexibility in which PSTs' reason conceptually about context problems in base-eight. The expectation that all participants reason in base-eight allowed them to experience mathematics in ways similar to how children experience learning whole number concepts and operations in base-ten. In the presented study, many PSTs reasoned in base-eight in ways similar to how children reason in base-ten (Beishuizen, 1993; Carroll & Porter, 1998; Huinker, Freckman, & Steinmeyer, 2003; Kamii, Livingston, & Lewis, 1993; Madell, 1985; Selter, 2002; Thompson, 1994). For example, the solutions that the PSTs presented were less procedural and were supported using a deep conceptual understanding of addition and subtraction. This conceptual understanding may assist these future teachers in understanding children's varied and unique ways of thinking about whole number addition and subtraction. In the end, the PSTs came to a surprising conclusion about their experience in base-eight and how it is similar to base-ten, as exemplified by the following classroom exchange:

Katie: I thought a lot about this on many nights that I can't sleep; base-ten.
Instructor: Okay, how many of you are thinking about this class outside of class? [most of the students raise their hands] I'm happy.
Katie: When you first said that we were going to learn base-eight, I thought it was going to be something it isn't, base-eight and base-ten aren't really different, they're alike, I don't really know how to explain it, they're like the same thing, just missing, they're not different … [Laughing] Does anybody know what I'm talking about? They're not like …
Sarah: They're the same process; they're exactly the same for both**.**

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