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What Do Croatian Pre-Service Teachers Remember from Their Calculus Course?

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Abstract

This paper reports a study on retention of core concepts in differential and integral calculus by examining the knowledge of two pre-service mathematics students. The study is conducted using a mixed method approach and the obtained data were analyzed using theory of three worlds of mathematics. The results showed that having good understanding of the concepts in the conceptual-embodied world can reflect on the long-term retention of mathematical knowledge in the proceptual-symbolic and formal-axiomatic world as well.

Keywords: calculus, retention, knowledge

Introduction

Teacher's knowledge of mathematics has been an area of great concern in the last two decades, since a conflict of opinion exists in the mathematics education community whether the abilities in higher level mathematics are the most important attributes for teaching or not. While some studies show that the competences in high level mathematics have profound effect on teaching (Hill & Ball, 2004), others argue that this is not sufficient to teach mathematics well and that teachers should also possess knowledge about teaching of mathematics and about students' misconceptions (e.g. Schommer-Aikins, Duell & Hutter, 2005). However, it is widely recognized that secondary mathematics teachers need deep knowledge of mathematics to be effective, and that the knowledge of content which will be taught in school is a crucial factor which influences the quality of teaching (Tatto et al., 2008) and is related to what and how students learn (e.g. Hill et al., 2005; Wilkins, 2002). Even though secondary mathematics teachers (mainly) have a major in mathematics, it has been documented that their content knowledge is generally lacking depth (e.g. Bryan, 1999).

All these concerns lead to a worldwide discussion about the appropriate type of mathematics education programs to be designed for future mathematics teachers (e.g. Zerpa et al., 2009). In Croatian mathematics secondary teacher education programs, mathematics method courses are designed to deal with the development of school mathematics, mainly focusing on the upper secondary school mathematics. Programs of mathematics teacher education in Croatia vary slightly from university to university, but in all of them pre-service mathematics teachers use their mathematical content knowledge, curriculum knowledge and knowledge of mathematics teaching in the classroom practice they have to complete in the final year of their studies.

In Croatian high schools, calculus is taught in the last year of schooling. Since calculus is also taught in the first year of mathematics teacher education programs, this topic is rarely addressed in mathematics method courses later on. In this paper we investigate the pre-service mathematics teacher's content knowledge regarding calculus. Specifically, we shall investigate the retention of calculus knowledge of two pre-service teachers in-depth, just before they started their classroom practice of teaching calculus in high school.

Theoretical Background

Three worlds of mathematics. To be able to describe students' retained knowledge, we will use Tall's (2008) theory of three worlds of mathematics which is based on perceptions of and actions on objects in the environment. Tall's classification is based on how mathematical concepts/objects are developed. This theory considers three aspects of mathematics, namely; geometric, symbolic, and axiomatic, and accordingly differentiates three worlds of mathematics: the conceptual-embodied world, the proceptual-symbolic world, and the axiomatic-formal world.

According to Tall, the world of *conceptual embodiment* grows out of our perceptions of the world and consists of things that humans can perceive and sense not only in the physical world but in their own mental world of meaning. It is based on perceptions of and reflections on properties of objects, seen in the real world and then imagined in the mind. Perceptions are described, defined and arguments are developed to formulate conclusions typical for Euclidean geometry, therefore, this world is concerned with human's visual–spatial imagery. In this world, language has a specific role. It is focused on describing properties of objects, and then on categorizing and defining.

This second world is the *proceptual-symbolic world*, used for calculation and manipulation in arithmetic, algebra, calculus and other mathematical areas. The development of the objects in this world begins with actions (such as pointing and counting) that are encapsulated as concepts using symbols which allow us to switch effortlessly from processes to be done to concepts to be though about. This combination of the symbol of a process and the concept produced from the process, Tall calls a procept.

The third world is *the formal-axiomatic world*, based on properties, expressed in terms of formal definitions and proofs. Here we are not working with familiar objects of experience, but with axioms that are carefully formulated to define mathematical structures (e.g. groups, fields, vector spaces, topological spaces) in terms of specified properties. Other properties are then deduced by formal reasoning (proofs) to build a sequence of theorems. The formal world arises from a combination of embodied conceptions and symbolic manipulations.

In Tall's theory of long-term learning, the development of knowledge is not linear and we can move back and forth between all worlds any number of times. This flexibility in moving between the worlds is the characteristic of good mathematical understanding.

Retention of knowledge. In this paper we will use the term retention in accordance with Sousa (2000) who defined it as the extent to which someone can successfully access and use the information from the long-term memory. Specifically for mathematics, investigating school children up to grade 10, Krutetskii (1976) has shown that not only the amount of what is forgotten, but also the type of forgotten knowledge, is related to student ability. Here, Krutetskii (ibid.) found that able mathematics school children retain around 85% of the generalized relations very well, even three months after they were learned. On the other hand, the long-term memories, especially those sustained over two months are a good indicator of what a student values (Davis & McGowen, 2001).

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Most studies of memory are oriented to memory for language and those of memory for mathematics are much less common. For instance, Davis & McGowen (2001) investigated pre-service mathematics teachers' retention of combinatorial problems. In their study, they reported on an antidote to procedural orientation to mathematics. Setting up a course with many related combinatorial problems, they asked students to reflect (as a homework task) on problems they solved during sessions and to make connections with earlier work. The study showed that along with the change in attitude towards mathematics, students experienced a corresponding growth in mathematical achievement and better long-term memory of learnt knowledge. Further, Narli (2011) investigated and compared the long-term effects of teaching Cantor set theory using a traditional and an active learning approach, respectively, on student's knowledge retention. Investigating pre-service mathematics teachers, the analysis of the data revealed that the students in the active learning environment (inquiry and problem based) showed better retention of almost all of the concepts related to Cantor set theory than the students in the traditional class.

Instead of solely focusing on retention, we argue that there is not necessarily a strict dichotomy between retention and non-retention, but that there is an intermediate state. Therefore, it is not the main goal of education that student immediately remembers all previously learnt, but to remember the knowledge that is being asked for after a hint. This, we argue, is just as good as retention and is in line with Karsenty (2002), who argues that recalling some mathematical knowledge is a reconstruction process that yields an altered version, in contrast to a reproduction, where details are coded in memory and they re-appear as so-called 'copies'. Thus, in our study, we used also hints to get better insight into the retention of calculus knowledge of pre-service teachers and, accordingly, we concentrated at the reconstruction and not on reproduction of knowledge.

Research questions

We formed the following research questions:

1. What do pre-service teachers remember from their calculus course several years after the course instruction and examinations?
2. How do they recall/reconstruct their knowledge?

Methodology

Participants. The study reported in this paper involved one male and one female pre-service mathematics teachers from one Croatian university. The male student will be called Mike and female student will be called Molly. Mike and Molly have been enrolled in the university almost five years. When this study took place, they were at the beginning of their student-practice in the secondary schools. This students-practice is a compulsory part of their mathematics teacher education study program. Unlike other pre-service teachers enrolled in the same year, Molly and Mike were willing to participate in the research study in which we examined their calculus knowledge and observed them in the class while teaching calculus concepts and procedures. In this paper we will focus on their calculus knowledge examined before their student-practice.

The calculus course Mike and Molly took was an enhanced mathematics course where elements of mathematical analysis are added to emphasize the theoretical background. Mike and Molly had to take written and oral exams in order to pass the calculus course. The written part of the exam contained calculus tasks, while the oral part examined their knowledge of mathematical theory. According to the grades obtained in their calculus course, Mike and Molly represented good students.

Mixed method approach. In this study we used a mix of research strategies; questionnaires supplemented with qualitative interview. Students were given two questionnaires. The first served as knowledge refreshment and a stimulus for knowledge reconstruction. The second questionnaire gave core information about students' retained knowledge. After the first questionnaire was filled in, we conducted an interview where we asked students to explain their work and thinking process. The interview covered additional questions that aimed at reconstruction of certain pieces of knowledge. Here hints were given to students which served as a cue for recall of earlier knowledge. According to Rittle-Johnson & Kmicikewycz (2008), generating information improves memory for that information and it also enhances the encoding of the item and relations between the item and the cue.

After knowledge activation with the first questionnaire, students were given a second questionnaire which they solved out loud. This questionnaire gave clearer information on how exactly students perceived the calculus concepts. According to Cohen et al. (2007), this kind of mixed method approach has significant advantages since using multiple methods provides triangulation of the data, hence help validating the conclusions.

Our paper is thus based on a case study of two students and the results may not be generalizable, however the case study can detect unique features that may otherwise be lost in larger scale data (Cohen et al., 2007). Also, the notion of generalizability, i.e. 'external validity', is here replaced by 'fittingness', "the degree to which the situation matches other situations in which we are interested" (Schofield, 1990, p. 207). On the other hand, Goetz and LeCompte (1984) use the notion 'translatability' to denote if the theoretical frames and research techniques are understood by other researchers in the same field, and the notion of 'comparability' to mean if a situation has been "sufficiently well described and defined that other researchers can use the results of the study as a basis for comparison with other studies addressing related issues" (Goetz and LeCompte, 1984, p. 228). In the case studies, detailed descriptions are important and necessary for others to be able to determine if the attributes compared are relevant (Kvale, 1996). Therefore we have provided a large amount of details about our study to make the process transparent.

Questionnaires on calculus concepts. In the educational context, the retention of knowledge is usually measured with cued recall and recognition. Recalling is usually examined using open-end questions, and recognition using true-false questions; on the other hand, multiple-choice questions combine recall and recognition (Custers, 2010). In this study we used open-end questions to get better insight into what student knows and what he of she had recalls. The mathematical notions chosen for the questionnaires reflected the intended learning outcomes of the calculus course and were constructed with the help of the teaching assistant of the course. The questions investigated various representations of derivative and integral notions across all three mathematical worlds. Some of the tasks were situated only in one particular world. For example, tasks that demanded performance of a certain procedure were connected only to the symbolic world, and those asking for definitions were connected to the formal world. Several tasks required moving between the worlds; in most cases between the symbolic and embodied world. All questions can be found in the Appendix.

The interviews were audio taped, transcribed and analyzed together with students' written work. The whole process lasted more than three hours per participant, and in that period participants were given refreshments and snacks. This way we wanted to ensure their complete activity and not losing the motivation for the rest of the process.

Results

In this section we will report on students' results from the questionnaires. We shall write Q*n* as abbreviation for "Question *n*". First we will describe results of the first questionnaire which served as knowledge refreshment, and after that we will report on Mike and Molly's results from the second questionnaire.

The first questionnaire

Mike's results

In Q1, belonging to the embodied world, Mike wrote that the geometrical interpretation of the derivative at some point was in fact the slope of the tangent line to the given curve at that point. He also made an appropriate figure of some curve with the tangent line at the point to support his written answer. However, Mike showed some difficulties with the derivative in the symbolic world. He did not apply the chain rule correctly in Q2. Although he could describe the chain rule, he used it incorrectly, missing some functions to be differentiated. His solution to Q3 was procedurally correct, but without answer formulation (which was expected since the question was written in textual form). In Q4, Mike established the connection between the symbolic and embodied world, and moved correctly back and forth to determine where f' and f'' change. Q5 was solved similarly as Q1. Mike embodied the concept of the integral using figures to support his answers. He also explained what would happen if the graph of the function was below the x-axis.

Although not asked, Mike gave a good definition of the definite integral using limits and integral sums, and then he realized that he gave incorrect answer in the case of Q6. In his written response, Mike tackled the formal world using Riemann sum to define antiderivative. However, Mike did not remember how to solve the Q8, and wrote $\ln(x^2)$ as the answer to given indefinite integral. The other proceptual-symbolic item, namely Q7, was correctly solved. Q9 required moving between the embodied and symbolic world, but was solved only in the embodied world. Mike made a drawing, but he was not able to write the definite integral needed to calculate the required area.

In order to try to reconstruct necessary pieces of knowledge, Mike was given hints in the interview for the incorrectly solved items. For instance, Mike was asked to look at the function in Q2, and to try to decompose it into its elementary parts, denoting arguments of those functions with different letters. After that he was asked to differentiate each part of the function and to explain if and how he would connect given pieces, and what these letters actually represent.

He was asked what the prefix "anti" suggests in the term antiderivative in Q6. In the case of the indefinite integral in Q8, he was asked to express the integrand in the different way, as a power of *x*. Lastly, it was pointed to Mike to observe his answer for Q5 in relation to Q9. Afterwards, Mike was asked to apply the answer to Q5 to solve Q9.

Molly's results

Molly experienced several difficulties in the embodied world. She was puzzled with Q1, since she did not understand what was meant by 'geometrical interpretation'. Instead of this, she wrote a partly correct formal definition of the derivative. Also, Q5 was left unanswered, and Molly pointed out once again that she did not understand what 'geometrical interpretation' means.

Molly had managed well in the symbolic world, performing appropriate procedures in Q2, Q7, and Q8. Molly explained: "I did many exercises of this kind in the calculus course, so I knew immediately what to do."

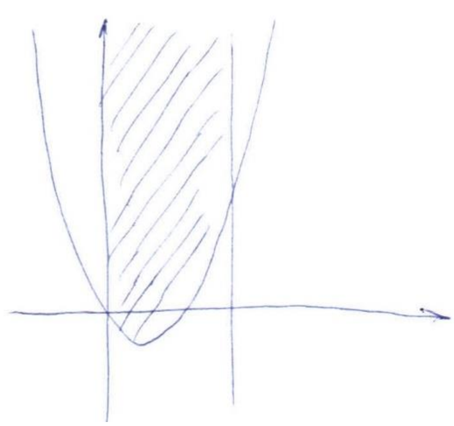
In Q3, Molly performed calculation and also wrote the answer as a full sentence. From her written work, and during the interview, she revealed she did not connect the term slope with the derivative, but that instead she recalled equation for the tangent line. When she was asked why she differentiated the given function to find slope of the tangent line, she explained: "I remembered the expression for the tangent line [pointing to her written work]. I saw there is a derivative so I used it."

This moment was used to trigger her retention and understanding about the geometrical representation of the derivative in a given point. She was asked what type of mathematical object the tangent line is (arithmetic, algebraic, geometrical) and what the derivative represented in this equation.

In Q4, Molly described that she traced the graph of the function with her finger, and that is how she decided where function increases or decreases However, she did not know whether her reasoning was correct or not, namely if *f*' is positive when the function increases or decreases, and vice versa. But she was convinced that the first derivative describes whether "the function is increasing or decreasing." She recognized the interval in which *f''* changes it's signs and she used her hand to gesture a switch in the shape of the graph.

Her reasoning was incorrect in Q9 where she concluded that the required area does not exist. Also she said that the area "goes to infinity", meaning that the area was not bounded from above (Figure 1). Here we see that she was not able to connect the definite integral with the concept of area.

Figure 1:
Molly's solution to Q9 in questionnaire 1



When it comes to formal world in Q6, Molly showed traces of remembering certain formalism. Here she wrote the fundamental theorem of calculus. She was asked about the relationship between functions F and f in the fundamental theorem and to try to express this relationship.

Interestingly, even after elaborations on her solutions and discussion about geometrical objects connected to the mentioned calculus concepts, Molly did not understand the term 'geometrical interpretation'. Also, she could not adequately determine the boundaries for the area item, even after repeated reading. It seemed that she could not include x-axis as the boundary to obtain closed shape for the area. Also, Molly said it was harder for her to remember integral calculus than derivative calculus.

Second questionnaire

Mike's results

In the second questionnaire, Mike showed very good long-term retention of the derivative concept; he was able to give the formal definition for the derivative and he was able to move very flexibly between the symbolic, embodied and formal worlds. Asked to explain his answer, Mike said:

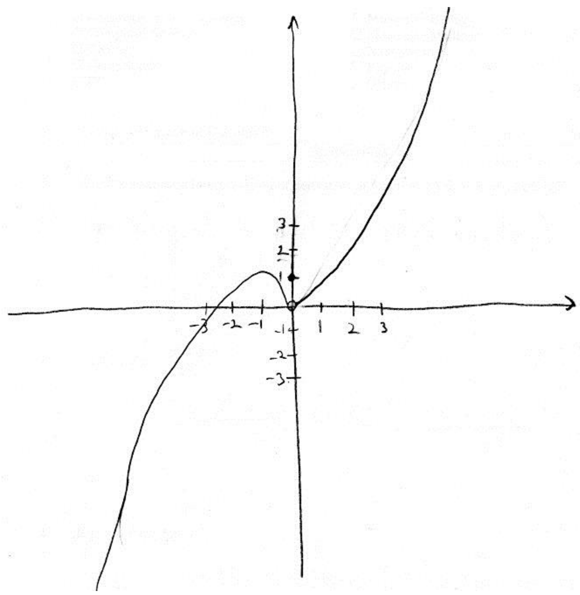
> "But this can be easily derived…Finding the derivative is same as calculating the slope of the tangent line in certain point on the graph of function… It says something about how graph behaves… is it raising or falling... We had a secant ... we watched a borderline case, when the secant will go into a tangent, and then we looked the ratio of differences; difference of function values in some point and the point where the tangent is, and the difference of these points [pointing to Δx]".

Q2 was solved in the embodied world where the differentiability of the function was characterized in terms of the shape of the graph, namely not having a tangent in the given point (3,0) and having a "cusp" here. The continuity of the function was also observed as the property of the graph, namely that the point (3,0) belonged to the graph of the function. However, Mike was not able to recall the definition of continuity: "I do not remember the epsilon-delta definition.... I do not remember that formulation. If I would have remembered this, I would use it."

In Q3, Mike applied the chain rule and here he helped himself writing down functions that constitute this composite function and making an arrow diagram to visualize in which order obtained functions are "inserting" into one another, i.e. are forming composition. So here Mike combined embodied concept of the function and symbolic rule. When Mike was explaining how he solved this question, he noticed one more way to differentiate this composite function: "Oh, well there is another way of solving this. I can use properties of logarithmic function and then I would get two logarithms...hm... I would skip this way one step in differentiation [pointing to the fraction]".

Q4 was solved not only symbolically, but also using notions from the embodied world. Besides calculating the first derivative and finding the stationary points, Mike used the number line to graphically represent position of calculated stationary point, to determine whether it is a point of extreme. Mike also demonstrated a good connection between the symbolic and embodied world in Q5. He made a satisfactory sketch where he included all required conditions (Figure 2). He explained that he could not recall what $f''(x) < 0$ means; whether this represented a condition for being concave upward or downward, but he helped himself with a quadratic function.

Figure 2:
Mike's solution to Q5 in second questionnaire



Mike's results also improved regarding integrals. He successfully solved all given problems. Here Mike distinguished the definite integral from the indefinite one across three worlds of mathematics in Q6. The definite integral was considered as a number whose numerical value represents area bounded by the graph of the given function on some interval, whereas the indefinite integral was characterized as a set of functions, and was connected with the anti-derivative. Here Mike referred to the previous questionnaire where he defined the definite integral using limits and integral sums. We see that Mike moved from symbolic into embodied and formal world to characterize these two notions. Similarly, Mike connected formal and symbolic worlds in the next item writing down the definition of the antiderivative and used it to solve Q7. In Q8, Mike used the concept-process relationship between integral and derivative and wrote the correct solution without any calculations. In Q9, Mike moved between the symbolic and embodied world, sketching down the area and performing the appropriate calculation.

Molly's results

Molly tried to give a formal definition of derivative of the function f in the given point x_0 using the quotient limit, but her written work was not accurate (Figure 3). Her solutions in both questionnaires and well as the explanation in the interview contained the same mistake in the numerator, which Molly was oblivious to when she explained her answer. Here she focused only on the difference and the limit, knowing that there must be minus sign in the limit expression, but it seemed she did not understand what the numerator and denominator represent.

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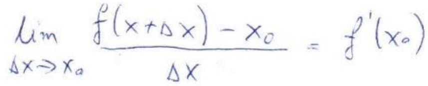
Figure 3:
Molly's formal definition of derivative



Q2 was solved in the embodied world, where Molly characterized the differentiability in terms of the shape of the graph, namely *x* = 3 being “stationary point”.
“It’s stationary point so it is no differentiable here. Here, down …. The function changes from in decreasing into increasing. And derivative here [in that point] is zero. See…here where the vertex is.?”

The continuity was also observed as the property of the graph, where a graph of the function can be drawn without lifting the hand from the paper, which is correct in the given case since the domain of the function is an interval. Molly said that she tried to remember the epsilon-delta definition for the continuity of the graph in the given point but without success: ”I know if something is less than delta, then other thing is less than epsilon. But I do not know what goes in the bracket [referring to the | |, i.e. absolute value]… I can’t remember.”

In Q4, Molly found extreme points in two ways. The first used only symbolic calculations where she combined the first and second derivative to determine stationary points and to examine if these stationary points are extreme points. In the second approach, Molly tackled the embodied world. Besides calculating the first derivative and finding the stationary point, she used a table for visualization of a turning point, i.e. where the function changes from increasing to decreasing and vice versa. Molly explained she that used the table to confirm her conclusion about extreme points. However, Molly was confused with Q5. Here she could not move from the symbolic into embodied world because she did not know what the second derivative represents graphically, and did not know how to represent the discontinuity. Also she was not able to graphically represent the given stationary point. She made a drawing where stationary point was actually zero point, possibly confusing the function value and the value of its derivative in a given point.

In Q6, Molly characterized definite and indefinite integrals only in the symbolic world, in terms of calculation where for the first one obtains a number, and for the second one obtains a formula. This reflected on Q7, where Molly obtained zero, and explained that she obtained the same formulas which cancelled. Q8 was solved using only the procedural aspect of the derivative and integral, i.e. by calculation. In the Q9, Molly moved between the symbolic and embodied world sketching a figure and then performing integration to find the required area. However, Molly made a mistake in her sketch which reflected on the final result. She misread $x = 2 - y$ as $x = 2$, obtaining a simpler area. If that were right, Molly’s solution would be correct. But when Molly was warned that she had misread the tasks, and the area became more complex, i.e. the area did not have only one curve as the lower boundary, Molly did not know how to deal with that.

Discussion

Mike showed a good level of retained knowledge and he successfully used given hints to reconstruct missing pieces in his network of knowledge. He moved back and forth between three worlds of mathematics, using connections between different representations of the same concept. The concept of derivative of the function was described with several representations (as slope of the tangent line, rate of change, process of differentiation, inverse of the integration, and as limit of the quotient of differences). These representations resided in different worlds of mathematics, and Mike successfully used them when moving between worlds and inside each world. Mike used the derivative as another function or as a number (derivative in a certain point) depending on the situation. According to Semb et al. (1996), the retained knowledge depends on the original learning, wherefore we believe that Mike embodied and symbolized derivative adequately when he was learning. We argue this enabled reconstruction of Mike's knowledge four years after the learning.

Similar can be said about the concept of the integral, where Mike distinguished the indefinite from the definite integral in an appropriate way, and used them according to the situation. The term 'integral' was separated in two meanings (definite and indefinite), and accordingly represented in all three worlds: as the limit of integral sums, as area, as set of functions, as process of calculation. These representations were mutually connected. However, the concept of continuity was characterized only in the embodied world. This concept did not appear in the first questionnaire wherefore we believe this knowledge could be not be entirely activated.

Molly's recall of calculus knowledge was not that good. She was tied to the proceptual-symbolic world, and referring to Semb et al. (1996), it seems she did not develop good understanding for the concept of the derivative and integral in various worlds when she was taking calculus. She remembered similar tasks she solved in her calculus course several years ago. Therefore, in the tasks where she had to move between the worlds, she did not reconstruct her knowledge, but rather reproduce it from the long term memory. For example, Molly did not know what the 'geometric interpretation of the definite integral' means although she solved tasks where this interpretation was the key. However, she retained very good knowledge of procedures for the differentiation and integration in the symbolic world. Molly was not flexible in moving between the symbolic and embodied worlds. For example, Molly used first and second derivatives of the given function to find points of extrema. Also she interpreted where f'' changes its sign looking at the graph of function, but she did not know how to interpret and sketch the graph of a function f with properties $f'(-1) = 0$ and $f''(x) < 0$ for $x < 0$. Also, she checked the continuity of the given function by looking at its graph, but did not know how to sketch a graph of a function with given point of discontinuity. Molly also had problems moving inside the embodied world. In one case she could find the required area and in two cases she could not. Also, she remembered only parts of formal definitions from the formal world.

Molly's retained knowledge indicates that she symbolized the concept of the derivative and the integral quite well, but did not embody and formalize those concepts when she was learning calculus. However, she remembered certain items from the course, e.g. the stationary point and the Fundamental theorem of calculus.

Conclusion

The aim of this study was not simply differing between the correct or incorrect answers. We wanted to investigate what students remembered form the calculus course taken several years ago, and how they recall that knowledge. Here we did not use the strict dichotomy between to remember or not to remember immediately after the question was posed, but we provided a sample of tasks to induce the process of recalling and used some hints to stimulate reconstruction process for forgotten items. We mainly used the second questionnaire to estimate what knowledge students retained many years after the course has taken place, however interesting findings emerged also from the first questionnaire. Even though the findings based of the two case studies cannot be generalized to a large population of pre-service teachers, we were able to detect some features that, we believe, would be hidden in large-scale studies, and could be tested on larger samples in future studies.

In Molly's case, the definitions were rote-learned, and were not fully integrated into a knowledge schema, therefore she remembered definitions incompletely or only partly correctly, and could not access fully the formal-axiomatic world. However, Molly retained a very good procedural aspect of knowledge from the proceptual-symbolic world, what is in contrast with many studies (e.g. Garner & Garner, 2000), where it was shown that procedural knowledge deteriorates faster with time and is remembered improperly if not connected with appropriate concepts.

The most interesting result of the study, in our opinion, is that Mike recalled formal definitions leaning on the embodied world. Mike reached more times into the embodied world to solve given problems. It seems that referring to the embodied world enabled him to reconstruct his knowledge. In some situations, Molly also used objects of the embodied world such as gestures to recall and explain her process of thinking. This indicates that building concepts in the embodied world is very beneficial for the learners in the tertiary education, especially for the pre-service mathematics teachers who will teach mathematics using objects from embodied world to create appropriate meaning for symbol manipulation. This is in line with study of Fothergill (2011), who states that more emphasis should be placed on connections to the secondary curriculum in preparation of future mathematics teachers in undergraduate calculus. On the other hand, using the embodied world in teaching calculus forms a motivational link to the study of advanced mathematics (Gray & Tall, 2007), which is often part of mathematics teacher education programs.

We do not mean hereby to argue that rigor in calculus courses should be diminished, since intuition is not always precise. Instead, we argue that it is necessary to spend a significant amount of time developing concepts in the embodied world, using them and returning back to the embodied world after the course moves on to the symbolic and formal world of mathematical analysis. Tall (2008) noted the amount of time spent in the embodied world is insufficient to build appropriate understanding of calculus concepts, and we support his observation, since focusing on the symbolism and formalism confuses pre-service mathematics teachers, and creates misconceptions in their knowledge.

Our results also indicate that correctly developed concepts in the embodied world help with correct usage and understanding of concepts in the symbolic and formal world (Mike), but that the fluency in the symbolic world does not help much in the embodied and formal world (Molly). As many other advanced mathematical concepts, e.g. vector spaces, can be represented in the embodied world, this should be taken into consideration in developing courses for pre-service teachers.

We believe that calculus tasks should connect all worlds to promote flexibility of mathematical thinking. For example, Q4 and Q5 in the second questionnaire show this: the concept of derivative is present in both, but the first of these two items does not require going into the embodied world and could be solved using only rote-learnt symbol manipulation, unlike the other, where the embodied world gives meaning to mathematical symbols. Something similar can be said about the Q2: often properties of differentiability and continuity are examined only in the symbolic world, with the aid of formal definitions, but also providing the graph brings the embodied world to the student and gives deeper meaning to the symbol manipulation.

The way mathematics teachers need to know and use mathematics is a bit different from the way pure mathematicians need and use mathematics. (Ball & Bass, 2000) Based on our findings, we would suggest more problems and tasks connecting the symbolic and embodied world, whenever it is possible, in calculus courses designed for pre-mathematics teachers. Another important issue that we wish to emphasize is the necessity of building many different representations of the same concept in various worlds. Tall (2008), in his theory of three worlds of mathematics, conjectured that connecting the embodied world with the symbolic and formal one is effective and enables flexibility in mathematical thinking. In our study we provided evidence that having such a network is important for the long-term retention of knowledge where a part of knowledge in one world enables recalling other connected parts.

References

Ball, D., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp.83-104). Westport, CT: Ablex Publishing

Bryan, T. J. (1999). The conceptual knowledge ofpre-service secondary mathematics teachers: How well do they know the subject matter they will teach? *Issues in the Undergraduate Mathematics of School Teachers: The Journal*. *1*.

Cohen, L., Manion, L., & Morrison, K. (2007). *Research Methods in Education (6th Edition)*. Routledge

Custers, E. J. F. M**.** (2010). Long-term retention of basic science knowledge: a review study. *Advances in Health Science Education*. *15*(1):109-128

Davis. G. & McGowen, M. (2001). What mathematical knowledge do preservice elementary teachers value and remember? In the *Proceedings of the 23 Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Snowbird, Utah.

Fothergill, L. (2011). Aspects of Calculus for Preservice Teachers. *The Mathematics Educator 21*(1), 23–31

Garner, B., & Garner,L.(2001). Retention of concepts and skills in traditional and reformed applied calculus, *Mathematics Education Research Journal*, *13*(3), 165-184.

Goetz J.P., & LeCompte M.D. (1984). *Ethnography and Qualitative Design in Educational Research*. London: Academic; 1984.

Gray, E., & Tall, D. (2007). Abstraction as a natural process of mental compression. *Mathematics Education Research Journal* 19 (2), 23–40

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371–406.

Hill, H., & Ball, D. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education*, *35*(5), 330-351

Karsenty, R. (2002). What adults remember from their high school mathematics? The case of linear functions, *Educational. Studies in Mathematics*., *51*, 117-144.

Krutetskii, VA. *The Psychology of Mathematical Abilities in Schoolchildren*, Chicago: The University of Chicago Press; 1976.

Kvale, S. (1996). *InterViews: An Introduction to Qualitative Research Interviewing*. London: Sage; 1996

Narli, S. (2011). Is constructivist learning environment really effective on learning and long-term knowledge retention in mathematics? Example of the infinity concept. *Educational Research and Reviews, 6*(1), 36-49.

Rittle-Johnson, B., & Kmicikewycz, A. O. (2008). When generating answers benefits arithmetic skill: the importance of prior knowledge. *J. Exp. Child. Psychol.* 101, 75-81.

Schofield, J.W . (1990). Increasing the Generalizability of Qualitative Research. In Eisner EW, Peshkin A, editors. *Qualitative Inquiry in Education*. New York: Teachers College. p. 201-232.

Schommer-Aikins, M., Duell, O., & Hutter, R. (2005). Epistemological beliefs, mathematics problem solving beliefs, and academic performance of middle school students. The Elementary School Journal. 105(3), 289-303.

Semb, G. B. and Ellis, J. A. (1994). Knowledge taught in school: What is remembered? *Review of Educational Research*, *64*, 253-286.

Sousa, D. A. (2000), *How the Brain Learns*, 2nd edition, Corwin Press: Thousand Oaks, CA.

Tall, D. (2008). The Transition to Formal Thinking in Mathematics. *Mathematics Education Research Journal*, 2008, 20 (2), 5-24

Tatto, M. T., Schwille, J., Senk, S., Ingvarson, L., Peck, R., & Rowley, G.(2012). *TeacherEducation and Development Study in Mathematics (TEDS-M): Policy, Practice and Readiness to Teach Primary and Secondary Mathematics. Conceptual Framework*, Teacher Education and Development International Study Center, Colledge of Education, Michigan State University, East Lansing, MI,

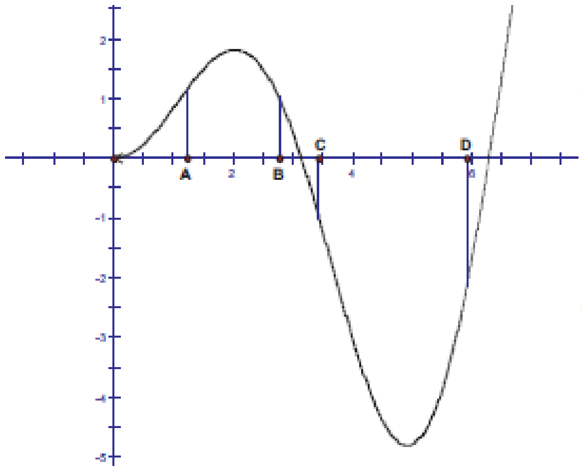
Wilkins, J. L. M. (2002). The impact of teachers' content knowledge and attitudes on instructional beliefs and practices. *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 24, 10.

Zerpa, C.**,** Kajander, A. & van Barneveld, C. (2009). Factors that impact preservice teachers' change in mathematical knowledge. *International Electronic Journal of Mathematics Education*, 4(2), 57-76.

Appendix

First questionnaire

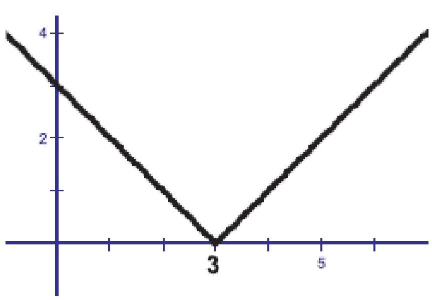
1. What is the geometric interpretation of the derivative of the function $f: \mathbb{R} \to \mathbb{R}$ at the point x_0?
2. Find $f'(x)$, if $f(x) = \sin^2 6x$.
3. What is the slope of the tangent line to the graph of the function $f(x) = (3x)^2$ at the point
   $x = 1$?
4. Look at the graph of the function f in the figure and answer the questions:
   a. What is the sign of f' on the interval $\langle 0, A\rangle$?
   b. Consider the following intervals $\langle 0, A\rangle$, $\langle A, B\rangle$, $\langle B, C\rangle$ and $\langle C, D\rangle$. In which of them does f' change its sign?
   c. Consider the following intervals $\langle 0, A\rangle$, $\langle A, B\rangle$, $\langle B, C\rangle$ and $\langle C, D\rangle$. In which of them does f'' change its sign?



5. What is the geometric interpretation of the definite integral $\int_a^b f(x)dx$, where f is bounded and $f(x) \geq 0$ for all $x \in [a, b]$?
6. Define the antiderivative of a function $f: \mathbb{R} \to \mathbb{R}$
7. Compute the integral $\int xe^x dx$. What is the most appropriate method for computing this integral?
8. Determine $\int \frac{dx}{x^3}$.
9. Find the area between the graph of the function $f(x) = x^2 - 2x$, the x-axis and lines $x = 0$, and $x = 3$.

Second questionnaire

1. Define the derivative of a function $f: \mathbb{R} \to \mathbb{R}$ at point x_0.
2. The function $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |x - 3|$. The figure below represents its graph.
   a. Explain whether or not this function is differentiable at $x = 3$.
   b. Explain whether or not this function is continuous at $x = 3$.



3. Differentiate $f(x) = \ln\left(\frac{1+x^2}{1-x^2}\right)$.

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4. Find the points of local extrema of the function $f: \mathbb{R} \to \mathbb{R}$ defined with $f(x) = x^2 e^x$.
5. Sketch the graph of a function f which satisfies the following conditions:
    a. f is discontinuous at $x = 0$ and $f(0) = 1$;
    b. $f''(x)<0$ for all $x < 0$ and $f''(x)>0$ for all $x > 0$;
    c. $f'(-1) = 0$ and $f'(x) \neq 0$ for $x \neq -1$.
6. Explain whether or not these two integrals are the same: $\int_a^b f(x)dx$ and $\int f(x)dx$.
7. If F is the antiderivative of f, what is $\int f(x)dx - F(x)$ equal to?
8. Given the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^2 + \sin x + 3$, find
    a. $(\int f(x)\, dx)'$;
    b. $\int f'(x)dx$.
9. Find the area above the x-axis bounded by the curve $y^2 = 4 - x$ and lines $y = 0$, $x = 2 - y$.