

CAPSTONE MATHEMATICS COURSE FOR TEACHERS

Conrad Van Voorst
SUNY College at Brockport
Education and Human Development
350 New Campus Drive
Brockport, NY 14420
cvanvoor@brockport.edu

Abstract

Three key features of a proposed capstone mathematics course for the preparation of teachers at the secondary level are: 1) teacher candidates examine concepts from the curriculum at a deeper level in the context of rich problems; 2) they investigate common themes in the math curriculum, such as shape and change, then collaborate on and present their research; and 3) they create portfolios of extensions and course reflections, products that require them to engage in higher-level, original thinking. Through this variety of assessments, the instructor is able to see future teachers of mathematics from multiple perspectives: their mathematical knowledge, their dispositions, and their beliefs about what mathematics is and how to effectively communicate the essential ideas. Participants' reflections indicate that those aspects of the course that have the greatest value for improving teaching are collaboration, problem solving, conceptual learning, and opportunities for exploration.

Introduction

Beginning teachers of mathematics often enter the profession lacking the knowledge they need to be able to effectively teach high school mathematics. The National Research Council (2002) speaks of “the gap both in mathematical content and in the ways the content is approached” that is created when students take four years of college math in a traditional program. In a recent publication by the Conference Board of the Mathematical Sciences (CBMS), the authors share the results of research done by the National Center for Research in Teacher Education indicating that “teachers who major in the discipline they teach are not more able than other teachers to explain fundamental concepts of their discipline. ...Teacher education programs that emphasize the underlying nature of the subject matter...more often result in knowledgeable, dynamic teachers with transformed dispositions and understandings of subject matter and pedagogy (CBMS, 2001, pg. 121). The purpose of this article is to propose a capstone mathematics course for pre-service teachers that will engage them in activities that lead to a deeper understanding of the mathematics they will be teaching. This course will be modeled after a similar course that is taught at the graduate level to in-service teachers who are working towards their Masters in Mathematics Education degree. The discussion will include the course goals and objectives, the course content, a sampling of class activities and performance tasks, and the course assessments. This paper will also present those key features of the course that have the potential of being effective in shaping prospective teachers' thinking about both mathematics and the teaching of mathematics. In-service

teachers in this graduate course have adapted many of the activities, assessments, and teaching approaches for use in classes with their own students. At the end of the semester, the teachers are asked to write a final course reflection paper in which they share their perceptions of the learning experience. Studies have shown that, “University teachers can improve the quality of student learning by determining how students perceive their unique learning situations” (Prosser and Trigwell, 1999). Common themes in teachers’ responses have been identified and considered to be useful for making decisions about key ideas to include when designing the comparable course for pre-service mathematics teachers.

The goals and outcomes of the described course for graduate students are very similar to those in the proposed undergraduate course for pre-service teachers. In response to the concerns of the National Research Council regarding beginning teachers, the capstone mathematics graduate course described in this article, is geared toward supporting new teachers who are already in the field through activities that lead to an understanding of the “big ideas” in the secondary mathematics curriculum. Shulman (1986) describes this kind of knowledge that is often demonstrated by experienced teachers, as *pedagogical content knowledge*. This course in many ways reflects the goals of a new capstone mathematics undergraduate course sequence recommended by CBMS (2001) “as an opportunity for prospective teachers to look deeply at fundamental ideas, to connect topics that often seem unrelated, and to further develop the habits of mind that define mathematical approaches to problems” (pg. 46). The capstone course should improve the preparation of teachers before they enter the teaching profession. Differences in the graduate and proposed undergraduate courses will probably surface in the way participants apply what they learn. Teachers who are in the graduate course who have the knowledge of how students think about mathematics and are able to immediately transfer what they learn into actual practice in their own classroom. Undergraduate, teacher candidates who take this course just before they do their student teaching, are in the midst of field experiences for their education methods classes, and are only able to observe the classroom practices of other mathematics teachers. By discussing the nature of mathematics and mathematical thinking in this course, prospective teachers should become more astute observers of those critical aspects of instructional practice that result in improved student understanding of mathematics.

One of the motivations for designing activities for this course was that prospective teachers should experience and ponder creative ways of introducing and communicating mathematical concepts. The goal is to engage teachers in the process of creating visual images that lead to an understanding of the more abstract mathematical concepts. Many of the problems and metaphors used are presented in the spirit of Bronowski’s belief that knowledge and imagination are “inseparable aspects of the intellectual experience” (Bronowski, 1978). For example, Bronowski shares the thinking that Newton used in the process of discovering the relationship involving the gravitational attraction between two massive bodies, “If I throw a ball, it will fall to the ground. If I throw it harder, it will fall a little further off. If I throw it harder still, it will fall still further off. I must be able to throw it just so hard that it falls exactly as fast as the horizon, and then it will go all the way around the world” (pg. 60). Another example, to help participants visualize the concept of infinity and broaden their perspective of infinity, is that they study Eli Maor’s inversion of a circle as presented in his book *To Infinity and Beyond* (Maor,

1987). To create images of this concept, Maor shares a quote from the *Auguries of Innocence*, by William Blake, “To see a world in a grain of sand, and a heaven in a wild flower, hold infinity in the palm of your hand, and eternity in an hour.”

As a result of taking this course, prospective teachers should feel the power and beauty of mathematics, should understand the big ideas and themes of the mathematics that they will be teaching in the curriculum, and should possess tools for communicating those ideas to their students in imaginative and meaningful ways. During the course, groups of participants are asked to think creatively as they present topics from the book *On the Shoulders of Giants*, to show evidence of new learning and of synthesis of information by submitting an extension.

Design of Course

Overview. This course addresses many of the topics covered in the secondary mathematics curriculum. The participants discover strategies for teaching these topics with the goal of developing students’ understanding of the underlying concepts. Activities used in this class reflect the goals of the NCTM Standards and the New York State MST Standards. Technology is used in the discussion of alternate ways of approaching problems and concepts. Due to a growing awareness that we should not teach mathematics in isolation from its applications, mathematical models are constructed from real data. Patterns are observed and represented numerically, graphically, and symbolically. To develop a greater appreciation for mathematics the participants read and present mathematical topics from the book *On the Shoulders of Giants* which focuses on various themes and patterns embedded in the secondary curriculum.

Student Mathematical Learning Goals. The process goals outlined below are introduced at the beginning of the course and continually applied and conceptualized by the students through the performance tasks. At the end of the semester, the students write a reflective essay in which they 1) explicate the degree to which they accomplished the objectives of the course and 2) identify their strengths, weaknesses, and areas of growth.

- Problem Solving and Modeling:
 - The student is able to model mathematical and applied problems – using mathematical functions to describe real-world behaviors.
 - The student is able to use an organized plan that leads to an efficient solution of a given problem.
 - The student is able to analyze the interrelationship between problems and situations, and use a variety of approaches to solve problems (algebraic and geometric, numerical and visual, or formal and experimental.)
- Conceptual Learning:
 - The student is able to select, access, and process mathematical concepts studied in this course.
 - The student is able to develop a lesson using a conceptual approach – focus on exploring and constructing meaning.
- Technology:

- The student is able to use the technology to explore math concepts and build mathematical intuition, and as tools to solve problems.
- The student makes mathematical conjectures based on the evidence provided by the calculator or computer.
- Mathematical Understanding:
 - The student is able to draw upon a variety of math skills, concepts, and procedures in building mathematical knowledge.
 - The student can reason inductively and deductively using the language of mathematics.
 - He or she can make conjectures based on observations and then use mathematical arguments to validate mathematical thinking.
- Active Learning and Research:
 - The student shows evidence of self-assessment and self-correction of work.
 - The student can independently explore mathematical concepts
 - The student collaborates by talking about mathematics with fellow students, by sharing in the collective decision-making process, and by communicating ideas about how to teach and think about mathematics from different perspectives and at different levels.

Kinds of Mathematical Content. In a report by the Conference Board of the Mathematical Sciences (CBMS) on the mathematical education of teachers the authors write that mathematics departments should support a “capstone course sequence for teachers in which conceptual difficulties, fundamental ideas, and techniques of high school mathematics are examined from an advanced standpoint” (CBMS, 2001, pg. 123). Usiskin, in a publication by the National Research Council (2002), refers to this as “teachers’ mathematics”. The types of mathematics content for the course proposed in this article and outlined below reflect their beliefs about what mathematics teachers need. Sample problems, activities, or investigations that follow each goal are designed to help teacher candidates think more deeply about the mathematics they will teach.

- Examine concepts:
 - *Discuss alternate representations of lines (standard, intercept, normalized, parametric) and parabolas. When would each be used? How can the algebraic, standard form be developed from the geometric definition?*
 - *Trace the historical development of concepts leading to the discovery of calculus*
 - *Investigate patterns of concepts in the math curriculum: (based on On the Shoulders of Giants - shape, dimension, uncertainty, quantity, and change)*
 - *Analyze the idea of a function as a transformation or mapping. Usually using x -axis to y -axis; now extend to 2-D set of points to another 2-D set. An interesting example is the inversion of a circle (Maor, 1987, pg. 8).*
- Integrate content taught in different units:
 - *Use a variety of problem solving approaches to find the number of oranges stacked in the form of a triangular pyramid consisting of n layers; 1 in first, 3 in second, 6 in third, and so on. Discuss the inductive and deductive approaches;*

solve the systems of equations using matrices or polynomial regression. Create a new similar problem (Stevenson, 1992).

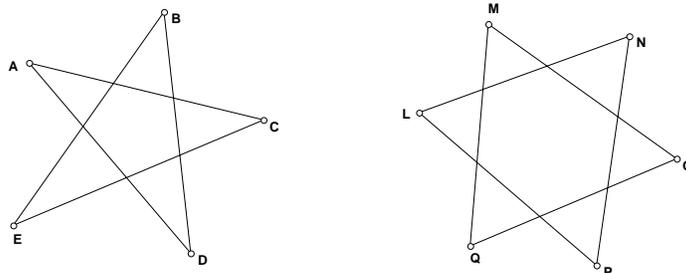
- *Use your graphing calculator and parametric equations to demonstrate the following motion: Road Runner is running at a constant speed of 50 ft/sec, and Wiley Coyote drops an anvil atop a 64 ft. cliff, when should Coyote drop the anvil?*



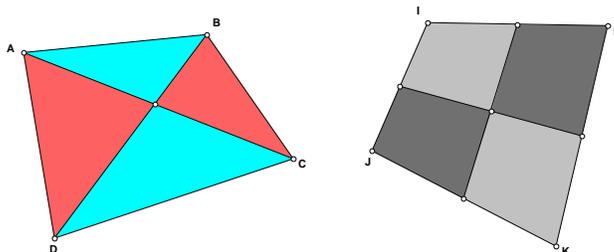
- Analyze the problem solving process using rich problems:
 - *Using the famous **Tower of Hanoi** problem, discuss how the recursive form more naturally represents the thinking process used in arriving at the solution. Obtain the explicit form by looking at the data, and show the recursive and explicit forms are equivalent.*
- Introduce concepts through real-world applications:
 - *A horse is tied to a tree by a rope that is 9 feet long. The tree has a radius of 3 inches. If the horse walks around the tree counterclockwise, describe the varying length of the rope that is free. Graph in the rectangular and polar coordinate systems. Which system is better for graphing this equation? Why?*
- Teach math modeling using open-ended problems:
 - *Design a container that holds 12 ounces or 355 ml of a liquid that uses the least amount of material. Create a list of factors that you would like to consider in making this decision. Support your choice of designs based on mathematics.*
- Make conjectures and write mathematical arguments:

Groups use Geometers Sketchpad (GSP) software to gain insights into solutions of geometry problems and then prove them using the theorems of geometry: problems from *Exploring Geometry with Geometer's Sketchpad* (1999)

- Find the general formula to determine the angle measure sum for an n -star polygon connecting every p th point (pg. 112).



- Section a piece of land in the shape of an irregular quadrilateral using two segments so that the four regions can be divided equally between 2 people (pg. 144).



Assessments. One of the goals of the course is to engage students in activities and performance tasks that help them conceptualize the objectives of the course. Bontempo and Morgan (2002) quote Edwards in discussing the effects of assessment type and practice on the quality of student learning: “assessments that reflect the need to understand concepts, inter-relate materials and problem-solve, presented often in the form of open-ended questions, essays, research projects, and problem-solving exercises, appear to encourage a deep approach to learning” (pg. 6). The conceptual approach to learning is used by students in the classroom, both in small- and large-group settings, as they solve routine and non-routine problems and discuss the kind of mathematical thinking and reasoning processes they used.

Problem sets and tests. On homework problem sets and tests, students demonstrate their knowledge of the facts, procedures, and concepts covered in the secondary school math curriculum and apply what they know in solving problems. To be able to solve the problems, they need to understand the meaning of the concepts and to represent them in multiple ways, can present geometric and algebraic solutions, to know how to use the technology, and can see the connections between mathematical ideas studied in the curriculum.

Portfolios. Assessment tasks that are student-generated and that require them to think at a deeper level about mathematics and the teaching of mathematics are included in a portfolio. Those areas that are assessed in the portfolio are students’ ability to: integrate

and apply knowledge in a meaningful way; collaborate effectively in researching and presenting a topic; use the process of invention; and reflect on their work. The assessments- -group research projects, extensions, and final course reflections essays- -measure both student performances and their perceptions.

For the *research projects*, the class is broken into groups of three or four each, and each group is assigned a chapter from the book *On the Shoulders of Giants*. Each author in the book is interested in presenting a fresh perspective on certain mathematical themes in the curriculum: dimension, quantity, uncertainty, shape, and change. The members of each group are to develop a plan for sharing these ideas with other members of the class; their plan could include discussions for integrating ideas into the present curriculum and some demonstrations, examples, or activities. This task requires students to look at the mathematics curriculum with new eyes.

The *extensions* require students to use their creative and critical thinking skills. The rubric used to evaluate their performance measures the degree to which they are inventive in either developing a meaningful application of a topic studied during the semester, or in creating a mathematical question or problem and providing mathematical support for their solution. Samples of students' extensions are:

- Shared discussion with eighth grade students involving the modeling of the growth of a flea population on a dog
- Presented a proof of the construction of the Golden Rectangle
- Felt incompetent with proofs, so did some research on different methods of proof and provided examples of each
- Used GSP to investigate "finding the largest triangle that can be inscribed in a circle with radius r ", then presented an elaborate proof that it is an equilateral triangle using principles of geometry, algebra, and trigonometry.
- Did research on exploration problems involving the use of the graphing calculator – calculus, parametric equations and polar coordinates, scatterplots and linear regression.
- Proved Heron's Theorem; then developed the equations of the hypocycloid and epicycloid (extension of proof of cycloid done in class)
- Extended the "dividing land" problem – explored the trisecting of a quadrilateral and proved the equal-area relationship
- Presented proof of a student presentation; "given any two points on $y=ax^2 + bx + c$, where $x_1 > 0$ and $x_2 > 0$, we can find a point of intersection at the y-axis by $y_3 = -ax_1x_2 + c$."

At the end of the semester the students are asked to write a *final course reflection* in which they share their perceptions of the course, their attitudes, motivations, and self-assessments of their learning. They discuss the degree to which they met the objectives of the course and those math topics, activities, or teaching methods that influenced, supported, or changed their thinking about mathematics or the teaching of mathematics. Students are asked to think more deeply about *how* they learn, what their *motivation* for learning is, and what tasks, problems, and teaching methods they can effectively use in their classrooms.

Analysis of Teachers' Perceptions From Course Reflection Papers

One of the purposes for gathering and investigating teachers' perceptions from this course is to determine which aspects of the course, from their perspective, are useful for improving their teaching. The investigation of their responses resulted in four major categories: collaboration, exploration, problem solving, and conceptual learning. For each category a sample of student responses follows and my observations concerning what the students will bring with them into their practice.

Collaboration.

- I thoroughly enjoyed the individual and group presentations...Both types of presentations are a good way for other teacher candidates to see how other people teach certain math topics.
- I sought out other teachers who I worked with to help me. It turned out to be more rewarding and brought me closer to my colleagues.
- I intend to use the concept of student mentoring in my own classroom. Not only will students understand math but also the mentor students will understand it to the point of mastering the concepts.
- Both my students and I have benefited from some of the lessons presented by my classmates in the course.
- Witnessing the collaboration of different philosophies of teaching was a neat experience that all future teachers should go through.

Observation on collaboration. Working together on presentations and problems helps build a sense of community. They learn from each other, mentor each other, as they strive for mastery of material and develop perspective. Learning through the use of cooperative groups forces students to take an active role in their learning.

Exploration.

- Learning the Geometer's Sketchpad made me rethink how to teach mathematics... using it to explore new concepts.
- As a student in this more open-ended course, I felt at ease and encouraged to explore...GSP adds a new realm of exploration with students for me.
- I have never seen so many exploration activities that relate to math at the high school level... it helped me see the importance of discovering concepts.
- This class supported my views of being a conceptual learner and teacher, and the importance of understanding math through exploration.
- I began using technology (calculator and GSP) for exploration in my class... I feel more comfortable implementing it in my own classroom.
- It has always been my goal to have students formulate math concepts, and I will renew my goal once again to not just lecture, but to further engage students in their own learning.
- This is the first year I have actually used the calculator in my classroom with my eighth graders to explore lines. As a result they have a much clearer understanding of slope and y-intercept.
- I am open to trying new things in my classroom. I have included more technology-based lessons where students are exploring different concepts. I have

also used more cooperative groups such as the Jigsaw activities. I have changed the way I used to think about mathematics, in the sense that I believe all students should be active learners.

Observation on exploration. Technology such as GSP and graphing calculators are useful for exploring math concepts. This exploration leads to a clearer and deeper understanding of the concepts and engages students in the learning process. It also adds a new dimension to their teaching.

Problem Solving.

- I have learned the importance of planning in solving real-world problems. Through practice I have developed more confidence in my ability. I will use many of the problems with my own students for enrichment.
- I was exposed to a teaching style that showed a variety of strategies to solve problems. It was invaluable to see this modeled. Also problems which would have been impossible in the traditional classroom can now be looked at using the calculators or GSP.
- I enjoyed the challenge of solving problems...learning the importance of asking the right questions.
- The Road Runner problem is excellent for motivating students. Students in my class created an extension to this problem with 2 cars going in opposite directions and meeting in a specified place.
- An improvement in my teaching is not only trying to deliver the concepts in different ways, but giving students several tools to solve problems...I had them first solve by trial-and-error and then they begged for a more efficient way using algebra.
- I enjoyed using regression... I am now using it in my classes to solve real-world problems involving compound interest.
- I have new ways of approaching problems that I would have never thought of before the course. Being a middle school teacher, it is easy to lose confidence in your ability to do higher-level math.

Observation on problem solving. They saw value in the teacher modeling problem-based instruction, and appreciated the fact that problems can be used to motivate student learning. They developed confidence in working through problems when they had more tools for approaching them.

Conceptual Learning.

- This course helped me realize that I never had a true understanding of concepts taught in calculus - the calculator in combination with problem solving and modeling made it happen for me.
- Every topic discussed in class seemed to challenge my way of thinking about the mathematics behind the concepts.
- The class gave me new insights into the development of concepts, especially calculus.

- I now understand the importance of patience and I have an appreciation for working through ideas and concepts - allowing students to make conjectures.
- As students we studied higher level math concepts and then were given the opportunity to discuss and model ways to introduce these topics to our own students in a student-centered approach.
- This course taught me about the underlying mathematical concepts; they were never clear to me in my undergraduate studies.

Observation on conceptual learning. Studying the development of concepts in the secondary curriculum challenged their way of thinking about math; the course gave them new ways to introduce math ideas to students. They saw the need for patience, allowing students to make conjectures and uncover math concepts embedded in problem situations.

Summary

The analysis of the behaviors and responses of teachers in this graduate course, isolated the important features, activities, and objectives that can improve the preparation of pre-service undergraduate mathematics teachers. A mathematics capstone course for pre-service teachers should include opportunities for participants to experience and talk about the kind of thinking students do when they learn mathematics. Activity-based and problem-based learning provides the right kind of setting for discussing the reasoning processes students use for constructing meaning of mathematical concepts, and for modeling effective mathematics teaching. Teachers gain deeper insights into the nature of mathematics, the usefulness of mathematics, and mathematical thinking when concepts are introduced and applied in the context of *rich* problems. One of the features of the course that has the greatest impact on their teaching is the use of the technology for exploration and experimentation. They realize that the technology is useful for making conjectures and that the principles of algebra and geometry are for justifying their answers.

As a result of completing the products that went into the portfolio, including projects, creative extensions, and final course reflections, teachers continue to develop their own mathematical perspective and a clearer understanding of the nature of mathematical knowledge. This kind of knowing, Magolda refers to as *contextual knowing*. Cross and Steadman (1996) indicate that “at this stage, perceptions of learning change from valuing thinking independently to valuing thinking through problems and integrating and applying knowledge in context in the light of evidence” (pg. 187). The portfolio as an assessment tool is useful to me as the instructor to obtain an accurate, specific picture of what individual teachers gain from the course. Through their perceptions, extensions, and presentations I am able to gain insights into their creativity, their ability to apply their knowledge of mathematics, their mathematical dispositions, and their ability to work collaboratively.

Teachers’ reflections on the effect of this course on their teaching and learning indicate the need for future research on the preparation of teachers in mathematics education. Future studies should focus on measuring the effectiveness of teacher preparation courses by evaluating the extent to which the teachers’ perceptions of the

changes in their teaching and thinking about mathematics transfer to actual changes in classroom practice.

References

- Bennett, D. (1999). *Exploring Geometry with The Geometer's Sketchpad*. Berkeley, CA: Key Curriculum Press, Inc.
- Bontempo, J. & Morgan, P. (2002). "Learning, Teaching, & Assessment: Overview of Key Literature on Learning, Teaching, and Assessment." Available: <http://www.training.wa.gov.au/resources>.
- Conference Board of Mathematical Sciences – CBMS (2001). "The Preparation of High School Teachers." *The Mathematical Education of Teachers*, 11, 121 – 145.
- Cross, K.P. & Steadman, M.H. (1996). *Classroom Research: Implementing the Scholarship of Teaching*. San Francisco, CA: Jossey-Bass Inc., Publishers.
- Maor, E. (1987). *To Infinity and Beyond: A Cultural History of the Infinite*. Boston, MA: Birkhauser, Inc.
- National Research Council. (2002). *Studying classroom teaching as a medium for professional development. Proceedings of a U.S.-Japanese workshop*. H. Bass, Z. Usiskin. & G. Burrill (Eds). Mathematical Sciences Education Board. Washington, DC: National Academy Press.
- Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15, 4-14.
- Steen, L.A. (1990). *On the Shoulders of Giants*. Washington, DC: National Academy Press.
- Stevenson, F.W. (1992). *Exploratory Problems in Mathematics*. Reston, VA: National Council of Teachers of Mathematics.