MAPLE Worksheet Number 10
Sequences and Series

Students often confuse the terms "sequence" and "series." "Sequence" refers to an ordered, usually infinitely long, list of numbers

$$\{ a_1 , a_2 , a_3 , a_4 , a_5 ,\ldots, a_i , \ldots \} .$$

As we saw before, the MAPLE command for generating a finite portion of a sequence is

seq(expression in terms of some index, index=starting integer value..ending integer value);

Use this command to generate the first 20 numbers in the following sequences

$$\text{a. } \{\frac{1}{i}\} \qquad \text{b. } \{\left(\frac{1}{i}\right)^2\} \qquad \text{c. } \{\left(\frac{1}{2}\right)^i\} \qquad \text{beginning with } i=1 .$$

"Series" refers to the sum of a sequence and the notation used is $\sum_{i=i_0}^{\infty} a_i$. Of course we can't add infinitely many numbers together so we do what we always do in Caluclus, we add more and more finitely many of the numbers together and hope for a limit. To this end we call $S_n = \sum_{i=i_0}^{n} a_i$ the n-th partial sum and define

$$\sum_{i=i_0}^{\infty} a_i = \lim_{n \to \infty} S_n .$$

The MAPLE command for generating a partial sum is

sum(expression in terms of some index, index=beginning integer value..ending integer value);

Use this command to generate the 20-th partial sums for a, b, and c above. Using cap S will display the notation. For example to display the notation and compute the sum of the first 20 terms of (a) use the command sequence

```
> Sum(1/i,i=1..20)=sum(1/i,i=1..20);
```

Now do the same thing for b) and c).

The series $\sum_{i=1}^{\infty} \frac{1}{i}$ is called the "harmonic" series. The series $\sum_{i=0}^{\infty} r^i$ is called the "geometric series with ratio r" and the series $\sum_{i=1}^{\infty} \frac{1}{i^p}$ is called a "p-series." The series in (b) is a p-series with p=2 and the series in (c) is a gerometric series with ratio $\frac{1}{2}$.

A series $\sum_{i=i_0}^{\infty} a_i$ is called "convergent with limit S" if $\lim_{n \to \infty} S_n = S$ and we use the notation $\sum_{i=i_0}^{\infty} a_i = S$, otherwise it is said to "diverge." For each of the harmonic, p-series, and geometric series above, determine if the series diverges or converges by computing first the n-th partial sum then taking its' limit as n -> ∞ by performing the following command sequences:

> $Sa_n := \sum_{i=1}^{n} \frac{1}{i}$

> $\lim_{n \to \infty} Sa_n$

> $Sb_n := \sum_{i=1}^{n} \frac{1}{i^2}$

> $\lim_{n \to \infty} Sb_n$

> $Sc_n := \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i$

> $\lim_{n \to \infty} Sc_n$

Now define Sp_n to be the n-th partial sum for a general p-series. In other words

> $Sp_n := \sum_{i=1}^{n} \frac{1}{i^p}$

Experiment by substituting various values of p into this expression and computing the limit, to determine for what values of p does the p-series converge, and for what values does it diverge. For example to compute with $p = \frac{1}{3}$ perform the following command sequence:

> $\mathrm{subs}\left(p = \frac{1}{3}, Sp_n\right)$

```
> limit(%,n=infinity);
```

Show sufficiently many examples to support your conclusion.

Similarily experiment with different values of r in the geometric series to determine for what values of r the geometric series converges and for what values it diverges.

Let's try to use MAPLE to "prove" our conclusion.

```
> with(student);
> Int(1/x^p,x)=int(1/x^p,x);
> Int(1/x^p,x=1..n)=int(1/x^p,x=1..n);
> simplify(%);
> Int(1/x^p,x=1..infinity)=limit(rhs(%),n=infinity);
```

Notice that this expression goes to infinity if the "effect" of n is in the numerator, i.e. $0 < 1 - p$, and goes to $\dfrac{1}{p-1}$ when the effect of n is in the denominator, i.e. $1-p<0$. What does this have to do with a p series? Well if we graph the function $x^{(-p)}$ from x=0 to x=n we observe that the n rightboxes are always under the graph while for x=1..n+1 the n left boxes are always above the graph. But the sum of the n rightboxes from 0 to n is the n-th partial sum of the p-series and the sum of the n leftboxes from 1 to n+1 are the n-th partial sum of the p-series. Thus the limit of the integral is infinite or finite if and only if the p-series is correspondingly infinite and finite. To illustrate this observation do the appropriate left and right box plots and sums for $x^{\left(\frac{1}{2}\right)}$ using 10 boxes and observe the relationship between the area under the graph and that included in the boxes.

```
> rightbox(1/x^(1/2),x=0..10,10);
```

Notice it is obvious from this picture that if

$\displaystyle\int_1^{\infty} \frac{1}{x^{\left(\frac{1}{2}\right)}}\,dx$ is finite then $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^{\left(\frac{1}{2}\right)}}$ is also finite.

```
> Int(1/(x^(1/2)),x = 1 ..
  infinity)=Limit(Int(1/x^(1/2),x=1..n),n=infinity);
> Int(1/x^(1/2),x=1..n)=int(1/x^(1/2),x=1..n);
> Limit(Int(1/x^(1/2),x=1..n),n=infinity)=limit(int(1/x^(1/2),x=1..n
  ),n=infinity);
```

Thus we have no information about whether or not $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^{\left(\frac{1}{2}\right)}}$ is finite or infinite. Continue with the following:

```
> leftbox(1/x^(1/2),x=1..11,10);
```

From this picture it follows that if $\displaystyle\int_1^{\infty} \frac{1}{x^{\left(\frac{1}{2}\right)}}\,dx$ is infinite then $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^{\left(\frac{1}{2}\right)}}$ is also infinite. Thus the sum is infinite from above. Of course MAPLE can arrive at either conclusion directly:

```
> Int(1/x^(1/2),x=1..infinity)=int(1/x^(1/2),x=1..infinity);
> Sum(1/i^(1/2),i=1..infinity)=sum(1/i^(1/2),i=1..infinity);
>
```