

MAPLE Worksheet Number13
Matrices and Vector Spaces

We recall that multiplication by an $n \times m$ matrix A is a "linear map" from the space R^m of vectors of length m to the space R^n of vectors of length n . Let's use MAPLE to demonstrate what we mean by a linear map. Define A to be an arbitrary 2×3 matrix, v and w to be arbitrary 3×1 matrices, (ie column 3-vectors) and let a and b be arbitrary scalars. Linearity means that

$$A(av + bw) = aAv + bAw.$$

Load linalg and execute the following commands to check this identity with MAPLE.

```
[ > A:=(matrix(2,3));
[ > evalm(A);
[ > v:=matrix(3,1);w:=matrix(3,1);
[ > evalm(v);evalm(w);
[ > evalm(A&*(a*v+b*w));
[ > leftside:=map(expand,%);
```

The MAPLE command `map(command,matrix)` is used to cause the specified command (expand in this case) to work on each entry of the indicated matrix. Now lets compute the other side of the identity and compare.

```
[ > aA:=evalm(a*A);bA:=evalm(b*A);
[ > rightside:=evalm(aA&*v+bA&*w);
[ > leftside-rightside;
[ > evalm(%);
```

The set of all vectors in R^m which go to the zero vector in R^n is called the "null space" of A . If A is nonsingular then the null space of A consists of only the zero vector. Why? Use MAPLE to demonstrate. A handy MAPLE command to use here is

$$\text{linsolve}(A,B).$$

This automatically tries to solve the equation $AX = B$ for the values of X . Apply it to try to find the null space for an arbitrary 2×2 matrix M . Note that MAPLE will assume that M is nonsingular, just as it did when we computed the formula for the inverse of such an M in worksheet 11.

Find the null space of the following matrix $K := \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{bmatrix}$.

Note that the element in the position (i,j) is $i+j$. So we can use one of our shortcut methods from Number 11 to define K

```
[ > K:=matrix(5,7,(i,j)->i+j);
So to solve  $KX = 0$  for  $X$  we use the command
[ > with(linalg):
[ > NK:=linsolve(K,matrix(5,1,0));
```

How many independent variables are used to describe the vectors in the null space of K ? This number is the "dimension" of the null space of K . It is also called the "nullity of K ." To actually find some vectors

in the null space of K we need only to choose values for these 5 independent variables and compute the resulting vector. List 5 different vectors in the null space of K gotten by choosing all but one of these variables equal to 0 and the remaining one equal to 1. For example define

```
[ > x[1]:=matrix(7,1,[5,-6,0,0,0,0,1]);
```

In similar fashion define x_2, x_3, x_4, x_5 . The null space of K is the "span" of the set $\{x_1, x_2, x_3, x_4, x_5\}$, in other words the set of all "linear combinations" of x_1, x_2, x_3, x_4, x_5 . To see that any combination of x_1, x_2, x_3, x_4, x_5 is in the null space of K compute and view $K(a x_1 + b x_2 + c x_3 + d x_4 + e x_5)$.

That everything in the null space is of this form is a consequence of the above solution. The set $\{x_1, x_2, x_3, x_4, x_5\}$ is called a "basis" for the null space of K . The MAPLE command `nullspace(matrix)` automatically computes a basis for the null space of the matrix. Try it on K . Does it yield the same basis we found above. Is the basis uniquely determined? Explain.

```
[ > NSK:=nullspace(K);
```

Now try our row reduction command on K .

```
[ > RRK:=rref(K);
```

How many independent row vectors are in the reduced echelon form of K ? The space spanned by these rows is called the "row space of K ." A basis for the row space of a matrix can be obtained immediately via the command `rowspace(matrix)`. Try it on K .

```
[ > rowspace(K);
```

Is it the same basis we obtained from our row reduction?

How many independent column vectors are there in the reduced echelon form of K ? The space spanned by the column vectors of the original K is called the "column space" of K . A basis for the column space of a matrix can be obtained immediately via the command `colspace(matrix)`. Try it on K .

```
[ > colspace(K);
```

Is the column space of K the same as the column space of the row reduced echelon form of K ? The moral of this story is that the row operations leading to `rref(K)` preserve the row space but not the column space. However the dimension of the column space is unchanged by the row operations and always equals the dimension of the row space. In the case of K the dimension of the row space equals 2 and is the same as that of the column space. The dimension of the row (or column) space of a matrix is called the "rank" of the matrix and is denoted by `rank(matrix)`, which is also the MAPLE command. Try it on K .

```
[ > rank(K);
```

Recall that the nullity of K is 5, we see that `nullity(K)+rank(K)=7` is the number of columns in K .

It is a fact that for any matrix M ,

$$\text{rank}(M) + \text{nullity}(M) = \text{the number of columns of } M.$$

Basically this says that if T is the linear transformation defined via multiplication by M then the dimension of the image of T plus the dimension of the null space of T equals the dimension of the domain of T . It follows that an $n \times n$ square matrix M is nonsingular, meaning the only vector it sends to 0 is the zero vector itself, if and only if `rank(M)=n` (or equivalently `nullity(M)=0`).

Now let's look at the basis for the null space of K . Recall it

```
[ > NSK;
```

Is this a "orthonormal" basis? In other words are these basis vectors all of norm (length) 1 and all perpendicular to each other. Recall that the length of a vector $v = (x_1, x_2, \dots, x_n)$ is given by the squareroot of the sum of x_i^2 . MAPLE has a command `norm`. Let's see if this is what we need. Perform the following command sequence.

```
[ > v:=vector([0,2,-3,1]);  
[ > norm(v);  
[ > sqrt(sum(v[i]^2,i=1..4));
```

Obviously this is not the norm we want. Experiment with some more vectors to see if you can determine what the command `norm` is computing. It's not too hard.

Looking in help we find optional statements can be added to the `norm` command. Try the following:

```
[ > norm(v,frobenius);
```

Let's use MAPLE to determine if `NSK` is orthonormal..

```
[ > NSK[1];  
[ > norm(%,frobenius);  
[ > dotprod(NSK[1],NSK[2]);
```

The first one we picked is not of norm 1, nor is it perpendicular to the second. There is a process we learned in linear algebra called the Gram Schmidt process which turns any set of basis vectors into an orthogonal basis. The MAPLE command for it is cleverly `GramSchmidt`(the set of vectors in question) . Lets try it on `NKS`.

```
[ > OSK:=GramSchmidt(NSK);
```

Check to see if these are all mutually orthogonal.

Next produce an orthonormal basis by dividing each of the above vectors by its norm.

```
[ > ONSK:={seq(OSK[i]/norm(OSK[i],frobenius),i=1..5)};  
[ >  
[ >
```

Check to see if these all have norm 1.