MAPLE Worksheet Number14

Recall that a non-zero vector V is an "eigenvector for the matrix M corresponding to the eigenvalue (lambda)" means

$$V = \lambda \quad .$$

This means that multiplication by M sends V to a vector parallel to V. Why do we require that V be a

number of components in M must equal the number of rows of M, the above identity implies that M is a square matrix. So the terms "eigenvector" and "eigenvalue" apply only to square matrices. The above equation is equivalent to

$$- \lambda I) \quad =[0]$$

where I is the identity and [0] is the zero vector, each of the appropriate size. Thus $_0$ is an eigenvector corresponding to eigenvalue λ_0 if and only if V_0 is in the null space of $M - \lambda_0 I$. It follows that any vector in the null space of $M - \lambda_0 I$ is an eigenvector associated with λ_0 . For this reason we call the null space of $M - \lambda_0$ the "eigenspace" of λ . Finally we observe that λ is an eigenvalue for M if and only if $M \quad \lambda \; I$ is singular, which is equivalent to

$$(M - \;_0 I) \qquad .$$

The matrix $M \quad \lambda$ is called the "characteristic matrix" of M, and the polynomial

$$\mathrm{p} \quad) = \quad (M - \quad I)$$

is called the "characteristic polynomial" of M. So the eigenvalues of M are exactly the roots of the

Use the above definitions and previously learned MAPLE commands to find the characteristic matrix, the characteristic polynomial, and all eigenvalues for the matrix

$$K \begin{bmatrix} 3 & 2 \\ & 0 \end{bmatrix}$$

Find an orthonormal basis for the eigenspace associated to each eigenvalue of K.

following command seqence and compare each result with that you obtained above.

```
> 
> K:=MATRIX([[3, 2], [2, 0]]);
  charmat(K,lambda);
```

```
> 
> factor(%);
  eigenvals(K);
```

To find eigenvectors for $\lambda \quad - \quad$ we need to compute the null space of $\quad - (\quad 1) I$

```
> nullspace(K+1);
```

```
> evalm(K&*[1,-2]);-1*[1,-2];
```

$$\lambda =$$

```
> nullspace(K-4);
```
To check this result compute
```
> evalm(K&*[2,1]);4*[2,1];
```
MAPLE has a command that computes eigenvectors directly. It is (well "duh") eigenvects(matrix). Try it on K.
```
> eigenvects(K);
```
What the heck does this mean? The -1 and 4 are the eigenvalues and the [1,-2] and [2,1] are the respective eigenvectors, but what are the 1 and 1 supposed to represent? Let's look at some more examples, and try to figure out what the "brilliant programers" intended to be represented by these extra numbers in the above expressions for the eigenvectors.

Compute the eigenvectors and corresponding eigenspaces for the following matrices. Remember, we are trying to figure out what the response to the eigenvects command represents! (Some day, maybe, I'll tell you how I came up with these.) Also, in each matrix, for each eigenvalue, determine the dimension of the corresponding eigenspace. (Be sure to name each of these suckers, you sure don't want to enter them more than once.)

1. $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -7 \\ 0 & 1 & 5 \end{bmatrix}$ 2. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ 3. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -6 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$ 5. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ 6. $\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$ 7. $\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{bmatrix}$

There are several important theorems about matrices, one is the
Caley-Hamilton Theorem: If P is the characteristic polynomial of a matrix A, then P(A)=0.

Verify the Caley-Hamilton Theorem for each of the above matrices.

Another important result is that the product of the eigenvalues equals the determinate of the matrix.

Verify this theorem for each of the above matrices. (Notice that a matrix has an eigenvalue of 0 if and only if it is singular to start with.)

Another result is that the dimension of the eigenspace of a particular eigenvector is no more than the multiplicity of that eigenvector as a root of the characteristic polynomial. Is this consistent with the above results? Explain.

The "trace" of a matrix is the sum of the diagonal elements and the MAPLE command for finding it is trace(matrix). It is a fact that the trace equals the sum of the eigenvalues. Verify this fact for each of the above matrices.

Another theorem is the following:
If $\lambda_1,...,\lambda_k$ are the distinct eigenvalues of an nxn matrix A with corresponding eigenvectors $x_1,..,x_k$, then $x_1,..,x_k$ are linearly independent. Verify this result for each of the above matrices.
A matrix M is called "diagonalizable" if there is an invertable matrix X such that $X^{(-1)} M X = \mathrm{D}$ for a diagonal matrix D.
A "mega-result" is the following:

An nxn matrix M is diagonalizable if and only if M has n linearly independent eigenvectors. Moreover, the diagonal matrix is composed of the eigenvalues, and the matrix X has the eigenvectors for columns.

The reason for this is actually quite straight forward. Let's consider one direction for a general 4x4 matrix M.
Suppose λ_1, λ_2, λ_3, λ_4 are the eigenvalues (not necessarily distinct) of M and v_1 , v_2, v_3, v_4 are corresponding linearly independent eigenvectors. Let V be the matrix whose columns are v_1 , v_2, v_3, v_4 .

So V=[v_1 , v_2, v_3, v_4] and $M\,V$=[$\lambda_1\, v_1$,$\lambda_2\, v_2$, $\lambda_3\, v_3$, $\lambda_4\, v_4$] ,

which is the same as

$$M\,V= V\begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_2 & 0 & 0\\ 0 & 0 & \lambda_3 & 0\\ 0 & 0 & 0 & \lambda_4\end{bmatrix},$$

which, since V is invertible, is the same as

$$V^{(-1)}\,M\,V = \begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_2 & 0 & 0\\ 0 & 0 & \lambda_3 & 0\\ 0 & 0 & 0 & \lambda_4\end{bmatrix}.$$

The same reasoning could be applied to any size square matrix as long as the corresponding V were invertible, which is equivalent to the eigenvalues being independent.

Notice what happens when we compute the power of the diagonal.

> $DM := \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

> DM^{10}

```
> evalm(%);
```

or even $\mathbf{e}^{DM}$

```
> exp(DM);
```

```
> evalm(%);
```

It follow that if M is diagonalizable then $M = V\, \mathrm{D}\, V^{(-1)}$ where D is the diagonal matrix.
So M^n=$V\, \mathrm{D}^n\, V^{(-1)}$and $\mathbf{e}^M = V\, \mathbf{e}^{\mathrm{D}}\, V^{(-1)}$. But D^nahd $\mathbf{e}^{\mathrm{D}}$ are easily computable as indicated above.

Work through the following sample diagonalization problem.

Let B be the 5x5 matrix with $a_{i,j}$=piecewise($i = j$, $i + j - 1$, $i + j + 1$) . Find all eigenvalues and corresponding
eigenvectors of B. Determine whether B is diagonalizable and, if so, find a diagonalizing matrix P and use it to diagonalize B.
Solution:

```
> B:=matrix(5,5,(i,j)->piecewise(i=j,i+j-1,i+j+1));
```

```
> P:=charpoly(B,lambda);
```

```
> EB:=fsolve(P,lambda,complex);
```

```
> EVB:=eigenvects(B);
```

The following command depends on the order in which MAPLE chose to list the eigenvectors and will probably need to be adjusted so that the indices are consistent with the order of your set of eigenvectors. You'll have to figure out what is right for your case.

```
> MB:=matrix([EVB[1][3][1],EVB[2][3][1],EVB[3][3][1],EVB[3][3][2],EV
  B[3][3][3]]);
```

Now we have to make the eigenvectors the columns of the diagonalizing matrix and perform the appropriate multiplication.

```
> transpose(MB)^(-1)&*B&*transpose(MB);
```

```
> map(simplify,evalm(%));
```

```
>
```