Elementary Algebra

(number4.mws)

In this exericse we study the basic properties of polynomials and rational functions. Some revelant MAPLE commands will be factor, expand, solve, fsolve, denom, numer, simplify, subs, ...

The first thing we need to learn is how to define a function. There are two ways, each with its own advantages and disadvantages. One way is as a symbolic statement as below:

> $p1 := x^2 - 1$

The other is as an operation as below:

> $p2 := x \rightarrow x^2 - 1$

Try the following command sequences to see some of the differences in how MAPLE treats these two definitions differently. As always record your observations and comments on the worksheet in text mode.

```
> p1;
> p2;
> p2(x);
> p2(t-3);
> p2(u^3-u^2+2);
```

This works just like we expect substitution to work because p2 is an operation on whatever is in the ( ). However since p1 is a symbolic expression we have some difficulties with substituting into it directly. Try the following command sequences.

```
> p1(x);
> p1(t-3);
> p1(u^3-u^2+2);
```

Notice the extra x in front. This is curious and not very reassuring. To substitute into a symbolic expression we must use the subs command.

```
> subs(x=t-3,p1);
> subs(x=6,p1);
> p2(6);
```

Now let's do some simple operations on this polynomial:

```
> factor(p1);
> factor(p2);
> factor(p2(x));
> expand(%);
>
> solve(%);
```

```
> fsolve(%%);
> solve(p2);
```

1. Explain the above results.

```
> solve(p1);
> solve(p2(x));
> solve(p1=2);
> fsolve(p1=2);
> solve(p2=2);
```

2. Explain the above results.

```
> solve(p2(x)=2);
> p3:=a*x^2+b*x+c;
> solve(p3);
```

3. Explain the above results.

```
> solve(p3,x);
```

4. Why did we need to include the ,x in the solve command in order to find the roots of p3?
5. What's the famous formula we get when we solve p3 for x?

```
> p4:=x->a*x^2+b*x+c;
> p4(x);
> solve(p4(t),t);
```

Expand , then factor each of the following expressions:

> $(x+3)(x-4)$

> $(a\,x+1)(b\,x+1)$

> $(x+I)(x-I)$

6. See if MAPLE knows a general formula for solving general cubic polynomials, i.e. Is there a 3rd order formula comparable to the quadratic formula? If so, exhibit it.
7. Why do you think high school students are never asked to remember the general formula for the roots of a cubic?

8. For the polynomials p_5 through p_{18}, solve for all exact roots whereever possible, find all approximate roots to 10 decimal places, factor each into linear factors.

> $p_5 := x^2 - 2$

> $p_6 := x^2 + 2$

> $p_7 := 2\,x^2 + 3\,x - 2$

> $p_8 := x^2 + x + 1$

> $p_9 := x^2 + (\sqrt{2} - 2)\,x - 2\sqrt{2}$

> $p_{10} := x^2 + (1 - \pi)\, x - \pi$

> $p_{11} := x^2 + (a + b)\, x + a\, b$

> $p_{12} := x^3 - x^2 - 2\, x$

> $p_{13} := 6\, x^3 + 11\, x^2 + 6\, x + 1$

> $p_{14} := x^3 + x^2 + x + 1$

> $p_{15} := x^4 - 1$

> $p_{16} := x^4 + 1$

> $p_{17} := 3\, x^4 + 13\, x^3 + 7\, x^2 - 17\, x - 6$

> $p_{18} := 6\, x^5 - 5\, x^4 + 4\, x^3 - 4\, x^2 - 2\, x + 1$

9. Explain why every odd degree polynomial must have at least one real root, while an even degree polynomial can have no real roots at all.

Perform the following MAPLE command sequence. Comment where appropriate.

```
> p:=x->5*x^3-5*x+1;
> r:=solve(p(x));
> r[1];
> simplify(r[1]);
> evalf(r[1]);evalf(r[2]);evalf(r[3]);
> p(r[1]);p(r[2]);p(r[3]);
> simplify(p(r[1]));simplify(p(r[2]));simplify(p(r[3]));
```

So these are the roots of p. However when we evalf them to get reasonable numbers they are all complex. This seems to contradict our statement about all odd degree polynomials having at least one real root. Try approximating the roots directly.

```
> fsolve(p(x));
```

Now we seem to have 3 real roots.

10. What appears to be going on?

Maybe drawing the graph will help. Plot the graph of p with appropriate choices for the range of x-values and the range of y-values. The MAPLE plot command is a "no brainer"

> plot(*expression*, $x = a \,..\, b$, $y = c \,..\, d$) where the choice of the y-range is optional. We could also include the optional command title=`name of the graph` to label our graph.

11. Find all the zeros of the polynomial $2x^4 + x^3 + x^2 + 1$.

12. Factor the polynomial $2x^4 + x^3 + x^2 + 1$ into linear factors.

13. What are the x-intercepts of the graph of the polynomial $2x^4 + x^3 + x^2 + 1$?

A "rational function" is defined to be the quotient of two polynomials. Define the following rational function:

$$> f := \frac{2\,x^5 + 5\,x^4 + 6\,x^3 + 6\,x^2 + 4\,x + 1}{4\,x^5 + 18\,x^4 + 24\,x^3 + 24\,x^2 + 20\,x + 6}$$

Some importrant features of a rational function are its x and y intercepts and its vertical asymptotes.
14. What determines the x-intercepts?
15. What determinres the vertical asymptotes?
Execute the following MAPLE command sequence:

```
> n:=numer(f);
> d:=denom(f);
> factor(n);
> factor(d);
```

16. What are the x-intercepts of f?
17. What is the y-intercept of f?
18. What are the vertical asymptotes of f? (Try plotting the graph of f to see this.)
19. Why does f have only one vertical asymptote when its denominator has 3 real roots?

```
> factor(f);
```

Note that factoring a rational function automatically simplifies it if possible.
20. What is the horizontal asymptote for f? (Try to see this from the graph of f.)
Execute the following command sequence.

```
> Limit(f,x=-1)=limit(f,x=-1);

> subs(x=-1,f);
```

21. Explain the results of the above command sequence.
22. Use your brain to compute the gcd(n,d) and lcm(n,d). (Hint: look at their factors.) Check your answer with MAPLE. (You may wish to use factor to compare your answers.)

MAPLE can preform some neat operations on polynomials and rational functions. For some examples try the following command sequence:

```
> degree(d,x);
> coeff(d,x,3);
> coeffs(d,x);
```

```
> nops(d);
> op(4,d);
> op(d);
```

```
> f;
> d;
> n;
> q:=quo(n,d,x);
> r:=rem(n,d,x);
> d*q+r;
```

23. How does this last result compare to n?
24. What do you think quo and rem are abreviations for?

```
> pff:=convert(f,parfrac,x);
```

This is really just another way to write f. To see that this equals f try the following.

```
> f-pff;
> simplify(%);
```

25. What do you think parfrac is an abreviation for?
26. Look at pff and determine $\lim_{x \to \infty} f$.
27. Use MAPLE to verify the result.

For demonstration purposes it is very handy to be able to construct polynomials with predetermined roots.

28. Use the expand command to construct a polynomial with roots $1, 2, 2, -\frac{1}{2}$, 1+I, and 1-I. (Here the "2,2" indicates that 2 is a "double root".)
29. Experiment with MAPLE to determine what number is represented by I. What is it?
30. What was the effect of having both the numbers 1+I and 1-I as roots? (See what happens if only one of them is a root.)