Limits in Calculus

(number6.mws)

The basic concept that defines "Calculus" it that of the "limit." The MAPLE command for computing the limit of some expression as some variable goes to some value is limit(expression, variable=value);
Define the function f1 and perform the indicated sequence of MAPLE commands.

> $f1 := \frac{x^3 - x^2 - 8\,x + 12}{x^2 + x - 6}$

```
> subs(x=4,f1);
> limit(f1,x=4);
> subs(x=2,f1);
```

> $\lim_{x \to 2} f1$

```
> subs(x=-3,f1);
```

> $\lim_{x \to (-3)} f1$

To see just what is going on here, factor the numerator and denominator of f1 and notice the common factors.

```
> factor(numer(f1));factor(denom(f1));
```

Define g as below.

```
> g:=factor(f1);
```

Does the function g equal the function f? To see that the answer is "no" perform the following:

```
> Limit(g,x=2)=limit(g,x=2);subs(x=2,g);Limit(g,x=-3)=limit(g,x=-3);
  subs(x=-3,g);
```

Note that g is defined and "continuous" at x=2 amd x=-3 while f1 is not.
Now check the limiting behavior of f1 and 1/f1 at infinity via the following sequence of MAPLE commands.

```
> Limit(f1,x=infinity)=limit(f1,x=infinity);
> Limit(f1,x=-infinity)=limit(f1,x=-infinity);
> Limit(1/f1,x=infinity)=limit(1/f1,x=infinity);
> Limit(1/f1,x=2)=limit(1/f1,x=2);
> Limit(1/f1,x=2,right)=limit(1/f1,x=2,right);
> Limit(1/f1,x=2,left)=limit(1/f1,x=2,left);
```

Compute the partial fraction decomposition of f1 and 1/f1 and see if these limits make sense.

```
> convert(f1,parfrac,x);convert(1/f1,parfrac,x);
```

Now plot the graphs of f1 and $\frac{1}{f1}$ to see if these limits make sense graphically.

```
> plot(f1,x=-3..3);
> plot(1/f1,x=-10..10,y=-10..10);
```

1. For each of the following functions compute the indicated substitutions and limits, and plot the graph

to see if the limiting behavior looks correct. Recall that a function is called "continuous a x=a" if its value at x=a equals its limit at x=a. In each case say whether the function is, or is not, continuous at the point in question.

> $f2 := \frac{-2\,x-4}{x^3+2\,x^2}$

> $\mathrm{subs}(x=0, f2)$

> $\lim_{x\to 0} f2$

> $\mathrm{subs}(x=2, f2)$

> $\lim_{x\to 2} f2$

> $f3 := (1+h)^{\left(\frac{1}{h}\right)}$

> $\mathrm{subs}(h=0, f3)$

> $\lim_{h\to 0} f3$

> $f4 := \frac{\sin(x)}{x}$

> $\mathrm{subs}(x=0, f4)$

> $\lim_{x\to 0} f4$

2. Similarly consider the behavior of $\sin\left(\frac{1}{x}\right)$, $x\sin\left(\frac{1}{x}\right)$, and $x^2\sin\left(\frac{1}{x}\right)$ near x=0 .

> $f5 := \mathrm{piecewise}\left(x=0, 0, \sin\left(\frac{1}{x}\right)\right)$

> $\lim_{x\to 0} f5$

a. Plot the graph of f5 and explain the MAPLE output for $\lim_{x\to 0} f5$.

> $f6 := \mathrm{piecewise}\left(x=0, 0, x\sin\left(\frac{1}{x}\right)\right)$

```
> subs(x=0,f6);
```

> $\lim_{x\to 0} f6$

b. On the plot of f6 include the graphs of the lines $y=x$ and $y=-x$.

> $f7 := \mathrm{piecewise}\left(x=0, 0, x^2\sin\left(\frac{1}{x}\right)\right)$

> $\lim_{x\to 0} f7$

c. On the plot of f7 include the graphs of the parabolas $y=x^2$ and $y=-x^2$.

Execute the following command sequences. They are intended to indicate more of the

MAPLE computer algebra features.

> $f8 := \frac{a\,x^3 + c\,x^2 - 1}{b\,x^3 - 2}$

> $\text{subs}(x = 1, f8)$

> $\lim_{x \to 1} f8$

> $\lim_{x \to \infty} f8$

> $f9 := \text{subs}(a = 0, f8)$

> $\lim_{x \to \infty} f9$

> $f10 := \text{subs}(b = 0, f8)$

> $\lim_{x \to \infty} f10$

3. Experiment with the function signum(x) and explain what signum(a) means in this expression.

Continue. First graph the function and see if you can "guess" the limit before computing with MAPLE.

> $f11 := \sqrt{x + 54} - \sqrt{x}$

> $\lim_{x \to \infty} f11$

> $f12 := \text{piecewise}(x < 0, x^3, x + 2)$

> $\lim_{x \to 0} f12$

> $\lim_{x \to 0-} f12$

> $\lim_{x \to 0+} f12$

> $f13 := x - \text{floor}(x)$

> $\lim_{x \to 4} f13$

> $\lim_{x \to 4-} f13$

> $\lim_{x \to 4+} f13$

> $f15 := \text{Heaviside}(t + 4)$

> $\lim_{t \to (-4)} f15$

> $\lim_{t \to (-4)-} f15$

> $\lim_{t \to (-4)+} f15$

Compute the following limits and comment on what the results tell us about the relative growth rates of the two functions involved with the quotient in each case.

> $\lim_{x \to 0} \frac{x}{\ln(x)}$

> $\lim_{x \to \infty} \frac{x}{\ln(x)}$

> $\lim_{x \to \infty} \frac{x^{100}}{e^x}$

> $\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}}$

Let's consider a multivariable limit. Investigate the limiting behavior with the limit and plot commands for the

function $\frac{x^2 - y^2}{x^2 + y^2}$. For example the MAPLE command for computing this limit at (1,2) is

```
> Limit((x^2-y^2)/(x^2+y^2),[x,y]=[1,0])=limit((x^2-y^2)/(x^2+y^2),{
  x=1,y=0});
```

Now compute the limit at (1,0), (0,1), and (0,0). Plot the graph and see if the limits make sense.

Recall the definition of the "difference quotient for a function G". Namely, if G (x) is a function then the difference quotient is defined as

$\frac{G(x+h) - G(x)}{h}$, or equivalently $\frac{G(x) - G(h)}{x - h}$. What does the difference quotient actually measure?

For each of the following functions compute its difference quotient and then compute the limit of the difference quotient as h goes to 0.

> $g1 := x \to (2\,x + 3)^5$

> $dqg1 := \frac{g1(x+h) - g1(x)}{h}$

> $\lim_{h \to 0} dqg1$

```
> factor(%);
```

Continue in the same way for g2, g3, and g4.

> $g2 := x \to \ln(x)$

> $g3 := x \to \frac{1}{x}$

> $g4 := x \to \sin(a\,x)$

Now compute the following limits of the difference quotient for a general expression G(x) .

```
> Limit((G(x+h)-G(x))/h,h=0)=limit((G(x+h)-G(x))/h,h=0);
> Limit((G(x)-G(h))/(x-h),h=x)=limit((G(x)-G(h))/(x-h),h=x);
```

In Calculus what do we call the limit of the difference quotient as h goes to 0 in the first or as h goes to x in the second.?