MAPLE Worksheet Number 8
Integration in Calculus via Area

We recall that $\int_a^b f\,dx$ is called the "definite integral of f from a to b" and is the "net area" between the graph of f and the x-axis between a and b, in other words the area above the x-axis minus the area below the x-axis. If the graph of f is made up of straight lines only, we can compute this definite integral using the formulae for the areas of triangles and rectangles. For example compute $\int_{-4}^{2} f\,dx$ for the function

```
> f:=piecewise(x<=-1,x+1,-x+4);
```

First plot the graph and then use the appropriate area formulae.

```
> plot(f,x=-5..5,discont=true);
```

The MAPLE command syntax for computing the definite integral of a function g with respect to the variable var between the values var=a and var=b is

int(g,var=a..b);

Try it to compute $\int_{-4}^{2} f\,dx$ for our f defined above. Notice that x is the integration variable. Convert to math symbolism.

```
> int(f,x=-4..2);
```

MAPLE has a very nice package designed to help students of Calculus visualize such area problems. It is loaded by the command

```
> with(student);
```

Many of these commands should have a familiar ring to them. Let's try the *box and *sum commands. After loading the student package, try the following commands.

```
> leftbox(f,x=-4..2,3);
> leftsum(f,x=-4..2,3)=value(leftsum(f,x=-4..2,3));
> leftbox(f,x=-4..2,7);
> leftsum(f,x=-4..2,7)=value(leftsum(f,x=-4..2,7));
> leftbox(f,x=-4..2,100);
> leftsum(f,x=-4..2,100)=value(leftsum(f,x=-4..2,100));
```

Notice how the more boxes we use the closer the answer agrees with the actual answer 6. Continue with the following command sequences using rightbox.

```
> rightbox(f,x=-4..4,7);
> rightsum(f,x=-4..2,7)=value(rightsum(f,x=-4..2,7));
```

Notice this is a lot closer than was leftsum using only 7 boxes. Let's use the average of the leftbox and rightbox idea. In other words we will use $\frac{\textit{area of the leftbox} + \textit{the area of the right box}}{2}$.

```
> (leftsum(f,x=-4..2,7)+rightsum(f,x=-4..2,7))/2=value((leftsum(f,x=
  -4..2,7)+rightsum(f,x=-4..2,7))/2);
```

This is pretty close. I seem to recall that the average of the leftbox area and the rightbox area is the area of the obvious trapezoid. Try the trapezoid command.

```
> trapezoid(f,x=-4..2,7)=value(trapezoid(f,x=-4..2,7));
```

I guess I remembered correctly. Now we try the middlebox approach.

```
> middlebox(f,x=-4..2);
```

This is interesting, is this picture correct? Middlebox is supposed to use the middle of the intervals to draw the top/bottom of the boxes.If your picture is not correct, shade in what should be the correct picture on your printout.

```
> middlesum(f,x=-4..2)=value(middlesum(f,x=-4..2));
```

Look at the correct picture and say why this "approximation" to the definite integral actually gives the exact answer. Apparently middlebox drew the picture incorrectly but middlesum computed the correct sum.

Of course the more interesting area problems occur when the graphs are not straight line segments, but are curves. For each of the following functions find the area of the region in the first quadrant, to the left of the line x=10, above the x-axis, and below the graph. Then use one of the "box" commands to plot the graph with the indicated area approximated by boxes. Use the corresponding "sum" and evalf commands to approximate the area to within 2 decimal places.

a. $3\sin(2x)$, b. $5x - 5x^3$, c. $\mathbf{e}^{\left(\frac{x^2}{100}\right)}$.

Notice if you want to find the total area, not the net area, between the graph of a function f and the x-axis between x=a and x=b, you simply compute $\int_a^b |f|\,dx$. Compute the total area for the functon $3\sin(2x)$ from x=0 to x=10.

Plot the graph of $|3\sin(2x)|$ and use one of the "box" commands to "shade" the area in question.

We can use this to estimate the total possible error in using the Taylor polynomial to approximate the values of a function near a point. For example to compute the error from using the 3rd degree Taylor polynomial to approximate sin(x) in the interval $-\frac{1}{10}$ to $\frac{1}{10}$ we compute the integral .

$$\int_{-.1}^{.1} |\sin(x) - \mathrm{taylor}(\sin(x), x = 0, 4)|\,dx$$

Compute this error, remember to convert the taylor expression to a polynomial.

```
> error:=sin(x)-convert(taylor(sin(x),x=0,4),polynom);
```

```
> Int(abs(error),x=-0.1..0.1)=int(abs(error),x=-0.1..0.1);
```

What's wrong with this answer?

Next try computing the exact error from $-\frac{1}{10}$ to $\frac{1}{10}$.

```
> Int(abs(error),x = -1/10 .. 1/10)=int(abs(error),x=-1/10..1/10);
```

Well Duh! Try evaluating the answer.

```
> Int(abs(error),x = -1/10 ..
  1/10)=evalf(int(abs(error),x=-1/10..1/10));
```

This is more like it. Too "confirm" this result use one of the numerical integration methods from above to see if you can approximate this same answer using $x = -.1$ to $x = .1$.

Explain why this should be the same as the following calculations

$$2 \int_0^h error\, dx$$

for h = 1/10. Try it and see.

```
> 2*Int(error,x=0..1/10)=2*int(error,x=0..1/10);
> evalf(2*int(error,x=0..1/10));
```

Now we consider some interesting examples in which MAPLE shows us some of its limitations. In each case plot the function and compute the definite integral for x=0 to x=10 using the int(f,x=a..b) command. For which of these does MAPLE give the correct answer and for which does it not. Explain your reasoning.

> $f_1 := \operatorname{floor}(\sin(x))$

> $f_2 := \operatorname{floor}(x^2)$

> $f_3 := x - \operatorname{floor}(x)$

> $f_4 := |\operatorname{floor}(x)|$

> $f_5 := \operatorname{floor}(|x|)$

The moral of this story is "When working with piecewise defined functions, be very careful to check the validity of the MAPLE results." We saw other examples in exercise number 5 where MAPLE had trouble with piecewise defined functions and floor is defined with infinitely many pieces.

Suppose we want to find the area of the finite regions bounded by f and g below.

> $f := x\, \mathbf{e}^{(-.5\, x)};\ g := -x^3 + 4\, x^2 - 3\, x$

Ploting their graphs on the same axes shows the regions in question.

```
> plot({f,g},x=-1..4,y=-2..3);
```

We need to solve for the intersection points.

```
> fsolve(f-g);
```

This yield only one of the three points, what about the other two. Looking at the graphs we see there should be intersection between x=-1 and x=1, between x=1 and x=2, and between x=2 and x=4. We can include bounds on x in the fsolve command.

```
> fsolve(f-g,x=-1..1); fsolve(f-g,x=1..2);fsolve(f-g,x=2..4);
```

Now we have them all. Use these to find the total area of the regions in question.

Similarily find the areas of the regions determined by the curves given below.

a) $y = 8\,x$ **and** $y = x^4$ b) $x^3 - y^2 = 0$ **and** $x + y^4 = 2$.

Let's try some "definite" integrals with "indefinite" limits of integration. For example compute using MAPLE $\int_1^a x^3\,dx$. In fact try $\int_a^b x^3\,dx$. This result should be suggestive. We'll return to it at the end of the exercise. But first let's consider some "improper integrals."   In particular we'll look at some integrals with one, or both, limits of integration being ∞ . Recall that

$$\int_a^\infty f\,dx \quad \text{is defined to be} \quad \lim_{b \to \infty} \int_a^b f\,dx \quad .$$

To demonstrate this definition perform the following sequences of MAPLE commands.

> $\int_1^a x^{(-2)}\,dx$

> $\lim_{a \to \infty} \int_1^a x^{(-2)}\,dx$

> $\int_1^\infty x^{(-2)}\,dx$

> $\int_1^a \frac{1}{x}\,dx$

>

> $\lim_{a \to \infty} \int_1^a \frac{1}{x}\,dx$

> $\int_1^\infty \frac{1}{x}\,dx$

What is the total area under the graph of $\frac{1}{x}$, above the x-axis, and to the right of x=1? What is the total volume of the object gotten from rotating this region about the x-axis? (WARNING, the "obvious" answer is wrong!)

If you have taken statistics, or probability, you will recognize the following function. Define

> $p := \dfrac{e^{\left(\frac{-x^2}{2}\right)}}{\sqrt{2\,\pi}}$

It's called the "normal distribution" and its graph is the usual "bell shaped" curve in statistics. The total area under its graph and above the x-axis is 1, the area under its graph and between x=-1 and x=1 is the portion of the total population that is 1 "standard deviation" form the mean. Plot the graph with suitable range on x to generate a nice bell shaped curve. Then compute the probability of being within 1 standard deviation from the mean. (You will have to use the evalf command to get an answer that makes sense. More about this in the next exercise on indefinite integration.) What is the probability of being within 2 standard deviations from the mean? Finally confirm that the probability of being in the population is 1 (ie the total area from $-\infty$ to ∞ is 1.

Execute the following MAPLE command sequences and see what you can conjecture.

> $\dfrac{\partial}{\partial t}\displaystyle\int_2^t x^3\,dx$

> $\dfrac{\partial}{\partial t}\displaystyle\int_{-6}^t x^3\,dx$

> $\dfrac{\partial}{\partial x}\displaystyle\int_a^x \sin(t)\,dt$

> $\dfrac{\partial}{\partial u}\displaystyle\int_q^u \sin(s^2)\,ds$

> $\dfrac{\partial}{\partial t}\displaystyle\int_{10}^t e^{(x^2)}\,dx$

One version of the FUNDAMENTAL THEOREM OF CALCULUS is the following:

If f is a continuous function then $\dfrac{\partial}{\partial x}\displaystyle\int_a^x f(t)\,dt = f(x)$.

Notice the result is independent of *a*. This can be shown to be equivalent to the following other version:

$$\text{If } \frac{\partial}{\partial x}F(x) = f(x) \quad \text{then} \quad \int_a^b f(x)\,dx = F(b) - F(a).$$

In this statement the function F(x)   is called the (indefinite) integral of   f(x) and denoted $\int f(x)\,dx$.
We will study techniques for computing indefinite integrals, and see how they are used to solve simple

initial valued differential equations in exercise number 8. In the mean time see if MAPLE knows the general statements of the Fundamental Theorem. Perform the following MAPLE commands, note that F(x) and a are undefined symbols. In each case convert the MAPLE input to the corresponding math symbol.

```
> Diff(int(F(x),x=a..t),t)=diff(int(F(x),x=a..t),t);
> Int(diff(F(x),x),x=a..b)=int(diff(F(x),x),x=a..b);
```